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A NOTE ON THE SEMIRADICAL OF A SEMIRING

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S. BOURNE has introduced a concept of a Jacobson radical of a semiring[1], and recently S. BOURNE and H. ZASSENHAUS have given a definition of a semiradical[2]. It is shown, that the semiradical contains the Jacobson radical of a semiring and both radicals do not coincide in general; [4].

But we shall note here that the semiradical coincides with the Jacobson radical. In this note we shall adopt the same terminologies and notations as in [4], but we shall recite them here for the sake of completeness.

DEFINITION 1. S is called a *semiring* if and only if

(i) A composition $+$ is defined in S such that $(S, +)$ is a commutative semigroup having an identity 0;

$$0 + a = a \qquad\qquad (a \in S).$$

(ii) A composition $\cdot$ is defined in S such that $(S, \cdot)$ is a semigroup.

(iii) Distributive laws hold;

$$(a+b)c=ac+bc, \quad a(b+c)=ab+ac.$$

DEFINITION 2. A subset I of S is called a *left* (*right*) *ideal* if and only if

(i) $i_1, i_2 \in I$ then $i_1+i_2 \in I$.

(ii) $i \in I$ then si (is) $\in I$ for all $s \in S$.

(iii) $0 \in I$.

We use the term a *two-sided ideal* of S as usual.

DEFINITION 3. If $i_1+x=i_2+x$ is solvable in S, i_1, i_2 are said to be *equivalent* and denoted as $i_1 \sim i_2$.

It is known that $\sim$ is an equivalence relation and the equivalence classes i^* represented by $i \in S$ form a semiring S^* according to the operations defined by

$$i_1^* + i_2^* = (i_1 + i_2)^*, \quad i_1^* \cdot i_2^* = (i_1 i_2)^*.$$

In the semiring S^* the cancellation law of addition holds. Therefore S^* can be imbedded into a ring $\bar{S}$ generated by S^* by the method given in [4].

DEFINITION 4. The right ideal I is said to be *right semiregular* if and only if for any i_1, i_2 of I there exist j_1, j_2 in I such that

$$(1) \qquad i_1+j_1+i_1j_1+i_2j_2=i_2+j_2+i_1j_2+i_2j_1.$$

DEFINITION 5. The *Jacobson radical* $R(S)$ of S is the union of all the right

1) Cf. [2].

2) Cf. [4].

semiregular ideals of S.

It has been proved that $R(S)$ is a two-sided ideal of S and is characterized as the maximal right ideal in which for every i_1, i_2 of $R(S)$ there can be found elements j_1, j_2 in $R(S)$ such that the equality (1) holds.

DEFINITION 6. The *semiradical* $\sigma(S)$ of a semiring S is the set of all elements i of S for which i^* is contained in the Jacobson radical $R(S^*)$ of S^*.

From the above definition, it follows immediately that $\sigma(S)$ is a two-sided ideal of S and $\sigma(S) \supseteqq R(S)$.

The following results are the known facts; [4].

(a) The semiradical of a semiring S is the maximal right ideal I of S in which for every pair of elements i_1, i_2 of I there exist j_1, j_2 in I and j in S such that

(2) $$i_1+j_1+i_1j_1+i_2j_2+j=i_2+j_2+i_1j_2+i_2j_1+j.$$

From (2) we obtain readily

(3) $$i_1+(j_1+j)+i_1(j_1+j)+i_2(j_2+j)=i_2+(j_2+j)+i_2(j_1+j)+i_1(j_2+j).$$

(b) The semiradical of a semiring S is the maximal right ideal I of S in which for any pair of i_1, i_2 of I a pair of elements j_1, j_2 of S can be found such that the equality (1) holds.

THEOREM. *The semiradical $\sigma(S)$ of a semiring S coincides with the Jacobson radical $R(S)$.*

PROOF. Put $I=\sigma(S)$. For any pair i_1, i_2 of I there exist j_1, j_2 in S such that

(1) $$i_1+j_1+i_1j_1+i_2j_2=i_2+j_2+i_1j_2+i_2j_1.$$

We multiply each term of the above equality by i_2 on the right,

$$i_1i_2+j_1i_2+i_1j_1i_2+i_2j_2i_2=i_2^2+j_2i_2+i_1j_2i_2+i_2j_1i_2.$$

Exchanging the sides of (1), we multiply each term of the resulting equality by i_1 on the right,

$$i_2i_1+j_2i_1+i_1j_2i_1+i_2j_1i_1=i_1^2+j_1i_1+i_1j_1i_1+i_2j_2i_1.$$

Adding the last two equalities and addint i_1+i_2 both sides of the resulting equality we obtain after rearrangement that

$$i_1+j_1'+i_1j_1'+i_2j_2'=i_2+j_2'+i_1j_2'+i_2j_1'$$

where $j_1'=i_2+j_2i_1+j_1i_2$, $j_2'=i_1+j_1i_1+j_2i_2$. As I is a two-sided ideal, we see that j_1', j_2' are in I. Hence from the definition of a Jacobson radical, we get $I \subseteqq R(S)$. Therefore we obtain $\sigma(S)=R(S)$.

We shall consider an example which is essentially the same as that of BOURNE and ZASSENHAUS; [4].

Let T_i be the semiring of all polynomials in the indeterminates x_i $(i=1, 2)$ with non-negative rational integral coefficients.

Let S be the semiring formed by the pairs (t_1, t_2) with $t_i \in T_i$ and the rules:

$(t_1, t_2)=(t_1', t_2')$ if and only if

(i) $t_2=t_2'$

and

(ii) $t_1=t_1'=0$ or $t_1t_1'\neq 0$,

addition and multiplication in S are defined as follows

$$(u_1, u_2)+(v_1, v_2)=(u_1+v_1, u_2+v_2),$$

$$(u_1, u_2)(v_1, v_2)=(u_1v_1, u_2v_2).$$

It is evident that all the elements $(t_1, 0)$ form an ideal A of S which consists of only two different elements $(0, 0)$, $(1, 0)$, and therefore A is not isomorphic to T_1. In this example $\sigma(S)=R(S)=A$.

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ON n-DIMENSIONAL FINSLERIAN MANIFOLDS ADMITTING HOMOTHETIC TRANSFORMATION GROUPS OF DIMENSION $> \frac{1}{2} n(n-1)+1$

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On groups of motions of Finslerian manifolds, H. C. Wang [7][1] showed that if a connected Finslerian manifold of dimension $n \neq 4$ admits an effective and connected group of motions of dimension $> \frac{1}{2} n(n-1)+1$ then the manifold is Riemannian and of constant curvature, and later this fact was proved by N. H. Kuiper and K. Yano [3] by a different method.

Recently, Gy Soós [6] treated homothetic transformations of Finslerian manifolds and showed that a Finslerian manifold with constant curvature $R \neq 0$ admits no homothetic transformation which is not a motion. Y. Nasu [5] also showed that a complete and connected Finslerian manifold of class C^1 admits a homothetic transformation which is not a motion then the manifold is Minkowskian. A similar theorem for a Riemannian manifold was already proved by S. Ishihara and M. Obata [2].

In the present paper, we shall study the structures of n-dimensional Finslerian manifolds admitting homothetic transformation groups of dimension $> \frac{1}{2} n(n-1)+1$. The results will appear in theorems of the paper. In the special cases where the manifolds are Riemannian our theorems coincide with already known theorems which were obtained by K. Yano [8] and the present author [1].

Let $\mathfrak{M}$ be an n-dimensional *connected* differentiable Finslerian manifold with fundamental metric function L. For each coordinate neighborhood U of $\mathfrak{M}$, L has the expression $L(u^i, \xi^j)$[2] which is defined on $u(U) \times R^n$, R^n being n-dimensional real number space and u the coordinate system of U. In the following we shall denote by ∂_i and $\dot{\partial}_i$ operators giving the derivatives of a function with respect to u^i and ξ^i respectively. We put

$$g_{jk} = \dot{\partial}_j \dot{\partial}_k F, \quad C_{jkl} = \frac{1}{2} \dot{\partial}_j \dot{\partial}_k \dot{\partial}_l F, \quad C^i_{jk} = g^{ia} C_{ajk} \left(F = \frac{1}{2} L^2\right),$$

$$\Gamma^{*i}_{jk} = \left\{ {i \atop jk} \right\} - C^i_{ja} G^a_k - C^i_{ka} G^a_j + C_{jka} G^a_b g^{bi},$$

$$\left\{ {i \atop jk} \right\} = \frac{1}{2} g^{ia} (\partial_j g_{ka} + \partial_k g_{ja} - \partial_a g_{jk}), \quad G^i_j = \frac{1}{2} \dot{\partial}_j \left(\left\{ {i \atop kl} \right\} \xi^k \xi^l \right)$$

and

$$R^{*i}_{jkl} = \partial_l \Gamma^{*i}_{jk} - \partial_k \Gamma^{*i}_{jl} - (\dot{\partial}_a \Gamma^{*i}_{jk}) \Gamma^{*a}_{bl} \xi^b + (\dot{\partial}_a \Gamma^{*i}_{jl}) \Gamma^{*a}_{bk} \xi^b$$
$$+ \Gamma^{*a}_{jk} \Gamma^{*i}_{al} - \Gamma^{*a}_{jl} \Gamma^{*i}_{ak},$$

1) See the Bibliography at the end of the paper.

2) Throughout the paper the indices take values $1, \cdots\cdots, n$.

where g^{ij} are elements of the inverse of the matrix $\|g_{jk}\|$[3], and we denote by a semi-colon an operator giving covariant derivatives of scalar and tensor fields with respect to Γ^{*i}_{jk}, for example,

$$T^i_{jk;l} = \partial_l T^i_{jk} - (\dot{\partial}_a T^i_{jk}) \Gamma^{*a}_{bl} \xi^b + T^a_{jk} \Gamma^{*i}_{al} - T^i_{ak} \Gamma^{*a}_{jl} - T^i_{ja} \Gamma^{*a}_{ak}.$$

We notice that, with respect to the coordinate neighborhood U, g_{jk}, C_{jkl}, $\dot{\partial}_l \Gamma^{*i}_{jk}$, R^{*i}_{jkl} and $T^i_{jk;l}$ are respectively components of tensor fields of $\mathfrak{M}$ and Γ^{*i}_{jk} is an expression of a linear connection of $\mathfrak{M}$.

A differentiable homeomorphism on $\mathfrak{M}$ is called a transformation on $\mathfrak{M}$ if its inverse is also differentiable. Given a transformation φ on $\mathfrak{M}$, take any coordinate neighborhoods U and U' such that $U \subset \varphi^{-1}(U')$. Then the φ maps U on $\varphi(U)(\subset U')$ and is expressed in terms of local coordinates as $u'^i = f^i(u^1, \cdots\cdots, u^n)$. If there exists a positive number α determined by φ only and

$$L'(u'^i, \xi'^j) = \alpha L(u^i, \xi^j) \quad (\xi'^i = \xi^a \partial_a f^i)$$

holds, then φ is called to be a homothetic transformation of $\mathfrak{M}$ and α the *associated number* of φ. If $\alpha = 1$, φ is called to be a motion or isometry.

Let G be a homothetic transformation group of $\mathfrak{M}$, that is, a Lie transformation group each element of which is a homothetic transformation of $\mathfrak{M}$. In the following we assume that G is *connected* and *effective*. To each φ of G there corresponds the associated number $\alpha(\varphi)$ and the correspondence α is a homomorphism of G into the multiplicative group of real positive numbers. If we denote by M the kernel of α then *M is a group of motions and*

$$\dim M = \dim G - 1$$

holds. We can also prove that *if G is compact G is necessarily a group of motions.*

We take a point p of $\mathfrak{M}$ and denote by G_p the isotropy group of p. Each φ of G_p induces a linear transformation $\tilde{\varphi}_p$ on the tangent space of p to $\mathfrak{M}$, where $\tilde{\varphi}_p$ is the differential of φ at p. The correspondence $\varphi \to \tilde{\varphi}_p$ is a linear representation of G_p onto the so-called linear isotropy group $\tilde{G}_p$ and, from the following Lemma 1, this linear representation is faithful. If we denote by M_p the kernel of the contraction of α to G_p and by $\tilde{M}_p$ the image of M_p under the linear representation then we have

$$\dim \tilde{M}_p = \dim G_p \text{ or } \dim \tilde{M}_p = \dim G_p - 1$$

according as the identity component of G_p is a group of motions or not.

H. C. Wang [7] proved that, *at any point p of $\mathfrak{M}$, a set of all the matrices $\|a^j_k\|$ satisfying the relation*

$$L(u^i_0, a^j_k \xi^k) = L(u^i_0, \xi^i)$$

for all ξ^i makes an orthogonal group up to a conjugation, (u^i_0) being the coordinates of p.

3) We assume that the matrix $\|g_{jk}\|$ has rank n.

Therefore the $\tilde{M}_p$ is necessarily a group of orthogonal transformations under a suiteble base.

Lemma 1. The linear representation $G_p \to \tilde{G}_p$ defined by $\varphi \to \tilde{\varphi}_p$ is faithful.

Proof. Let φ be an element of G_p such that $\tilde{\varphi}_p$ is the identity element of $\tilde{G}_p$. Taking a cubic coordinate neighborhood V of the point p covered by local coordinates (u^i) $(|u^i|<a)$, since φ leaves invariant the point there exists for the V a suitable small neighborhood $W(\subset V)$ covered by local coordinates $(u^i)(|u^i|<b, b<a)$ such that $\varphi(W)\subset V$. Denoting by $u'^i=f^i(u^1, \cdots, u^n)$ the expression of φ in terms of local coordinates, we have

$$f^i(0, \cdots, 0)=0 \quad \text{and} \quad (\partial_j f^i)_{u=0}=\delta^i_j, \tag{1}$$

$(0, \cdots, 0)$ being the coordinate of p. For arbitrary but given values ξ^i, we have

$$\partial_j \partial_k f^i = \underset{\xi}{H}{}^i_{jk}\left(u, \frac{\partial f}{\partial u}\right) \equiv \Gamma^{*a}_{jk}(u, \xi)\partial_a f^i - \Gamma^{*i}_{bc}(u', \xi')\partial_j f^b \partial_k f^c \tag{2}$$
$$(\xi'^i \equiv \xi^a \partial_a f^i)$$

which holds on the demain: $|u^i|<b$.

We consider a system of partial differential equations

$$\partial_j \partial_k u'^i = \underset{\xi}{H}{}^i_{jk}\left(u, \frac{\partial f}{\partial u}\right) \tag{3}$$

with n unknown functions $u'^1, \cdots, u'^n$. (2) shows that (3) has the solutions $f^i(u^1, \cdots, u^n)$ with the initial conditions (1). On the other hand, functions $u'^i=u^i$ are clearly solutions of (3) which have the same initial conditions. Therefore these solutions must coincide, more precisely, for a suitable positive number $c(<b)$ we have $f^i(u)\equiv u^i(|u^i|<c)$ and consequently $\partial_j f^i=\delta^i_j$. Thus we see that there exists a neighborhood $U(\subset W)$ of p such that, for any point q of U, $\varphi \in G_q$ and $\tilde{\varphi}_q$ is the identity element of $\tilde{G}_q$.

We consider a set N of points q of $\mathfrak{M}$ such that $\varphi \in G_q$ and $\tilde{\varphi}_q$ is the identity element of $\tilde{G}_q$. It follows immediately from the above argument that N is open in $\mathfrak{M}$. On the other hand, we can easily prove that N is closed in $\mathfrak{M}$. From the assumption that $\mathfrak{M}$ is connected we have $N=\mathfrak{M}$. Therefore φ leaves invariant each point of $\mathfrak{M}$ and is the identity element of G_p because G is assumed to be effective.

A tensor field T of $\mathfrak{M}$ of any type, for instance, of type (1, 2) is said to be invariant under a homothetic transformation φ if the relation

$$T'^i_{ab}(u', \xi')\partial_j f^a \partial_k f^b = T^a_{jk}(u, \xi)\partial_a f^i \ (\xi'^i \equiv \xi^a \partial_a f^i)$$

holds with respect to the local coordinates of any coordinate neighborhoods U and U' such that $U \subset \varphi^{-1}(U')$.

Here we notice the following. For a homothetic transformation φ we have easily

$$R^*(u', \xi') = \alpha(\varphi)^{-2} R^*(u, \xi) \ (\xi'^i \equiv \xi^a \partial_a f^i)$$
$$(R^* \equiv R^{*a}{}_{jka}\, g^{jk}),$$

from which we have a fact obtained by Gy Soós [6]: *if a scalar field of $\mathfrak{M}$ whose local*

expression is R is constant and not equal to zero then there exists no homothetic transformation of $\mathfrak{M}$ which is not a motion.

Lemma 2. Let φ be a homothetic transformation of $\mathfrak{M}$ which is not a motion and leaves invariant a point p of $\mathfrak{M}$ and T a tensor of type (r, s) at the point. If the components $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}(u, \xi)$ of T are homogeneous of order t with respect to the variables ξ^i and $r-s-t\neq 0$ holds, (u_0^i) being the coordinates of p, then the T is a zero-tensor, that is,

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}(u_0, \xi)=0$$

for all ξ^j.

Proof. Since φ is a homothetic transformation which is not a motion, the associated number $\alpha(\varphi)$ is not equal to 1. Therefore we assume without loss of generality that $\alpha(\varphi)<1$. For any given positive integer m, the associated number of the homothetic transformation φ^m is $\alpha(\varphi)^m$ and

$$L(u_0, \underset{(m)}{\alpha^j_k}\xi^k)=L(u_0, \xi^j)$$

holds for all ξ^j, where we have put

$$\underset{(m)}{\alpha^j_k}=\alpha(\varphi)^{-m}(\partial_k \underset{(m)}{f^j})(u_0)$$

and $\underset{(m)}{f^i}(u)$ is the expression of φ^m around p. The T being invariant under φ^m we have for any given values ξ^j

$$\begin{aligned}(4)\qquad &\underset{(m)}{\alpha^{a_1}_{j_1}}\cdots\underset{(m)}{\alpha^{a_s}_{j_s}}T^{i_1\cdots i_r}{}_{a_1\cdots a_s}(u_0^i, \underset{(m)}{\alpha^j_k}\xi^k)\\ &=\alpha(\varphi)^{m(r-s-t)}\underset{(m)}{\alpha^{i_1}_{a_1}}\cdots\underset{(m)}{\alpha^{i_r}_{a_r}}T^{a_1\cdots a_r}{}_{j_1\cdots j_s}(u_0^i, \xi^j)\end{aligned}$$

Now we can assume without loss of generality that the matrix $\|\underset{(m)}{\alpha^i_j}\|$ is orthogonal. Therefore the matrices $\|\underset{(m)}{\alpha^i_j}\|(m=1,2,\cdots)$ make a sequence in the full orthogonal group $O(n)$ and there exists a subsequence $\|\underset{m_l}{\alpha^i_j}\|(l=1,2,\cdots)$ which converges to an orthogonal matrix $\|\beta^i_j\|$ because $O(n)$ is compact.

When l tends to infinity, the left hand side and the second factor of the right hand side of the relations obtained from (4) by replacing m by m_l converge to

$$\beta^{a_1}_{j_1}\cdots\beta^{a_s}_{j_s}T^{i_1\cdots i_r}{}_{a_1\cdots a_s}(u_0^i, \beta^j_k\xi^k)$$

and

$$\beta^{i_1}_{a_1}\cdots\beta^{i_r}_{a_r}T^{a_1\cdots a_r}{}_{j_1\cdots j_s}(u_0^i, \xi^j)$$

respectively. If $r-s-t$ is positive, $\alpha(\varphi)^{m_l(r-s-t)}$ tends to zero and we have

$$T^{i_1\cdots i_r}{}_{a_1\cdots a_s}(u_0^i, \beta^j_k\xi^k)=0,$$

from which

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}(u_0^i, \xi^j)=0$$

by virtue of the arbitrariness of ξ^j. If $r-s-t$ is negative, $\alpha(\varphi)^{m_l(r-s-t)}$ tends to infinity and we have

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}(u_0^i,\ \xi^j)=0.$$

We notice that tensor fields whose components are respectively $g^{ij}F_{;j}$, $C^i_{jkm;l}$, $\dot{\partial}_l\Gamma^{*i}_{jk}$ and $R^{*i}{}_{jkl}$ are invariant under any homothetic transformation. Hence we have

Theorem 1. If a Finslerian manifold admits a homothetic transformation which is not a motion and fixes a point then the above cited tensor fields of $\mathfrak{M}$ are zero at the point.

Lemma 3. If $n\neq 4$ and

$$dim\, G>\frac{1}{2}n(n-1)+1$$

then G is transitive.

Proof. Take any point p of $\mathfrak{M}$ such that

$$dim\, G_p>dim\, G-n.$$

Then we have

$$dim\, \tilde{M}_p\geqq dim\, G_p-1>dim\, G-n-1\geqq\frac{1}{2}(n-1)(n-2).$$

According to a theorem of D. Montgomery and H. Samelson [4], in an Euclidean space of dimension $n\neq 4$, the full rotation group cannot have a subgroup K such that

$$\frac{1}{2}(n-1)(n-2)<dim\, K<\frac{1}{2}n(n-1).$$

Therefore we have

$$dim\, \tilde{M}_p=\frac{1}{2}n(n-1)$$

because $\tilde{M}_p$ is an orthogonal transformation group under a suitable base, and consequently $dim\, G>dim\, G_p$. From $dim\, G>dim\, G_p$ and

$$dim\, \tilde{M}_p=\frac{1}{2}n(n-1),$$

it follows that G is locally transitive at the point, that is, the orbit of G passing through p contains a suitable small neighborhood of p.

On the other hand, G is clearly locally transitive at any point q of $\mathfrak{M}$ such that

$$dim\, G_q=dim\, G-n.$$

If we take an arbitrary but fixed point p_0 of $\mathfrak{M}$ and denote by N the orbit of G passing through p_0, then N is open and closed in $\mathfrak{M}$. Consequently, from our assumption that $\mathfrak{M}$ is connected, we have $N=\mathfrak{M}$ which proves our lemma.

Theorem 2. Let $\mathfrak{M}$ be a connected Finslerian manifold of dimension n. Then the maximum dimension of effective and connected homothetic transformation groups of $\mathfrak{M}$ is $\frac{1}{2}n(n+1)+1$.

In fact, let G be an effective and connective homothetic transformation group of $\mathfrak{M}$ and assume that

$$dim\, G > \frac{1}{2} n(n+1)+1.$$

Then we have at a point p of $\mathfrak{M}$

$$dim\, \tilde{M}_p \geqq dim\, G_p - 1 \geqq dim\, G - n - 1 > \frac{1}{2} n(n-1)$$

which is a contradiction. Hence

$$dim\, G \leqq \frac{1}{2} n(n+1)+1.$$

It follows from the following example that $\frac{1}{2} n(n+1)+1$ is the maximum: a group of all homothetic transformations in an Euclidean space. This example is trivial, but the appositeness of this will be understood from the following Theorem 3.

H. C. Wang [7] showed that in a connected Finslerian manifold of dimension $n \neq 4$ there exists no effective and connected group H of motions such that

$$\frac{1}{2} n(n-1)+1 < dim\, H < \frac{1}{2} n(n+1).$$

By using this fact we have easily

Theorem 3. Let $\mathfrak{M}$ be a connected Finslerian manifold of dimension $n \neq 4$. Then $\mathfrak{M}$ admits no effective and connected homothetic transformation group G which is not a group of motions, such that

$$\frac{1}{2} n(n-1)+2 < dim\, G < \frac{1}{2} n(n+1)+1.$$

Theorem 4. Let $\mathfrak{M}$ be a connected Finslerian manifold of dimension n and G an effective and connected homothetic transformation group of $\mathfrak{M}$. (a) if

$$dim\, G = \frac{1}{2} n(n+1)+1$$

then $\mathfrak{M}$ is locally Euclidean and (b) if $n \neq 4$ and

$$dim\, G = \frac{1}{2} n(n-1)+2$$

then $\mathfrak{M}$ is Minkowskian.

Proof. First we prove (a). The group G is not a group of motions. In fact, if it is true, we have at a point p of $\mathfrak{M}$

$$dim\, \tilde{M}_p = dim\, G_p \geqq dim\, G - n = \frac{1}{2} n(n-1)+1$$

which is a contradiction. Therefore G contains a subgroup of motions whose dimension is equal to $\frac{1}{2} n(n+1)$. From a theorem of H. C. Wang [3], [7], $\mathfrak{M}$ is necessarily Riemannian. It is clear that G_p of each point p of $\mathfrak{M}$ is not a group of motions. Therefore, from Theorem 1, the tensor field of $\mathfrak{M}$ whose components are $R^{*i}{}_{jkl}$ is zero on $\mathfrak{M}$ and hence $\mathfrak{M}$ is locally Euclidean.

Next we prove (b). At each point p of $\mathfrak{M}$, G_p is not a group of motions. In fact, if it is true, we have

$$dim\ \tilde{M}_p = dim\ G_p \geqq dim\ G - n = \frac{1}{2}(n-1)(n-2)+1,$$

from which

$$dim\ \tilde{M}_p = \frac{1}{2}n(n-1)$$

because $n \neq 4$. Since G is transitive from Lemma 3, we have

$$dim\ G = \frac{1}{2}n(n+1)$$

which is a contradiction. By using Theorem 1, the tensor fields whose components are respectively $g^{jk}F_{;k}$, $\dot{\partial}_j \Gamma^{*i}_{kl}$ and $R^{*i}{}_{jkl}$ are zero on $\mathfrak{M}$. From the fact the second and third tensor fields are zero on $\mathfrak{M}$ it follows that each point of $\mathfrak{M}$ has a suitable coordinate neighborhood U of the point on which the connection parameters Γ^{*i}_{jk} are zero. Therefore, from $g^{jk}F_{;k}=0$, we have $L_{;k}=\partial_k L=0$ on U and consequently L do not contain the variables u^i. This proves that $\mathfrak{M}$ is Minkowskian.

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ON THE DEGREE OF ELECTRON DEGENERACY IN THE STELLAR INTERIOR

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Abstract

The degree of electron degeneracy in the stellar interior, decreases outward in the region where the transfer of energy is governed by radiative or conductive process and is constant in the region where the transfer of energy is governed by convective process.

§ 1. Adiabatic invariability of ψ

We shall consider the system consisting of N identical particles in a volume V. Then the Gibbs free energy G is

$$G=N\mu=H-ST=U+PV-ST \quad \cdots\cdots\cdots \quad (1)$$

where μ is the chemical potential, H the enthalpy, S the entropy, T the temperature, U the internal energy and P the pressure in our system.

We shall define a quantity ψ by the equation

$$\psi=\frac{\mu}{kT} \quad \cdots\cdots\cdots \quad (2)$$

where k is the Boltzmann constant.

Now we shall show that the differential coefficient of ψ with respect to T is zero during an adiabatic process, if $N=$*constant* and the forces between particles are *negligible* in our system. To do this, by using (1) and (2), we calculate the following:

$$\left(\frac{d\psi}{dT}\right)_{ad}=\left(\frac{\partial\psi}{\partial T}\right)_{S}.$$

By making use of the well-known thermodynamic relations

$$\left(\frac{\partial H}{\partial T}\right)_S=\frac{V\,C_p}{T\left(\frac{\partial V}{\partial T}\right)_p} \quad \text{and} \quad C_p=\left(\frac{\partial H}{\partial T}\right)_p,$$

we obtain the following results:

$$\left(\frac{d\psi}{dT}\right)_{ad}=-\frac{1}{NkT^2}\left\{V\left(\frac{\partial H}{\partial V}\right)_p-H\right\}$$

$$=-\frac{1}{NkT^2}\left\{V\left(\frac{\partial U}{\partial V}\right)_p-U\right\} \quad \cdots\cdots\cdots \quad (3)$$

In any thermodynamic system where the forces between particles are negligible, the following virial relation holds in all the statistics (that is, Fermi–Dirac, Bose–Einstein and Boltzmann statistics):

$$U = \frac{3}{2} PV \qquad (4).$$

Substituting (4) in (3), we obtain the final result

$$\left(\frac{d\psi}{dT}\right)_{ad} = 0 \qquad (5).$$

Thus we arrive at the following theorem:
During an adiabatic process, the value of ψ does not depend on the temperature if the number of particles does not vary and the forces between particles are negligible small quantity.

§ 2. Adiabatic change of Fermi gas or Bose gas

In the system consisting of Fermi particles or Bose particles, the equation of state is

$$P = \frac{2}{3} A\, I_{3/2}(\psi)\, T^{5/2} \qquad (6)$$

$$\frac{\rho}{m} = B\, I_{1/2}(\psi)\, T^{3/2} \qquad (7),$$

where ρ is the density, $A = \frac{2\pi g\,(2m)^{3/2} k^{5/2}}{h^3}$, $B = \frac{2\pi g\,(2mk)^{3/2}}{h^3}$,

g the statistical multiplicity arising from the spin of a particle, m the mass of a particle, h the Planck constant, and

$$I_l(\psi) = \int_0^\infty \frac{x^l\, dx}{e^{x-\psi} \pm 1},$$

the positive sign referring to the Fermi-Dirac, and the negative sign to the Einstein-Bose systems.

Differenciating logarithmically the equation (6) and (7) by T, and using the relation (5), we obtain the following theorem:

When the system, composed of the mechanically independent identical Fermi-Dirac particles or Einstein-Bose particles, changes adiabatically, the following relations always hold, provided that the number of particles does not vary during this process:

$$\left(\frac{d \log P}{d \log T}\right)_{ad} = \frac{5}{2} \qquad (8)$$

$$\left(\frac{d \log \rho}{d \log T}\right)_{ad} = \frac{3}{2} \qquad (9)$$

and hence

$$\left(\frac{d \log P}{d \log \rho}\right)_{ad} = \frac{5}{3} \qquad (10).$$

Formally these relations are just the same in the case of perfect gas composed of the Boltzmann particles.

§ 3. Adiabatic change of the partially degenerate gas

In the partially degenerate gas system, the gas pressure is the resultant of both degenerate electron pressure and nondegenerate ion gas pressure, while the radiation pressure is so small that its proportion to the total pressure is negligible. Thus the equation of state of the partially degenerate gas is

$$P = \Phi(\psi)\, T^{5/2} \qquad (11)$$

$$\rho = B_1 F_{1/2}(\psi)\, T^{3/2} \qquad (12)$$

where $\Phi(\psi) = A_1 \left\{ \frac{2}{3} F_{3/2}(\psi) + \frac{\mu_e}{\mu_i} F_{1/2}(\psi) \right\}$, $A_1 = \frac{4\pi (2m)^{3/2} k^{5/2}}{h^3}$,

$B_1 = \frac{4\pi \mu_e m_H (2mk)^{3/2}}{h^3}$, m_H = the mass of a hydrogen atom,

$\frac{1}{\mu_e}$ = the free electron number per unit atomic weight,

$\frac{1}{\mu_i}$ = the ion number per unit atomic weight,

and $F_l(\psi) = \int_0^\infty \frac{x^l\, dx}{e^{x-\psi} + 1}$.

Using the equation (11), (12) and (5) we obtain the following theorem:

When the partially degenerate gas system, in which Coulomb iteraction force is negligible, alters adiabatically, the following relations hold if the number of free electron does not vary during this process.

$$\left(\frac{d \log P}{d \log T} \right)_{ad} = \frac{5}{2} \qquad (13)$$

$$\left(\frac{d \log \rho}{d \log T} \right)_{ad} = \frac{3}{2} \qquad (14)$$

and hence

$$\left(\frac{d \log P}{d \log \rho} \right)_{ad} = \frac{5}{3} \qquad (15).$$

§ 4. The degree of electron degeneracy in the stellar interior

At present it is generally believed that the free electron gas is degenerate in the interior of the white dwarf or of the very late red dwarf[(1)], or in the central core of the giant star[(2)]. In general the number of free electrons depends upon the effect of ionization of atoms by pressure or by temperaure,.

However, under the physical conditions that must prevail in stellar interiors, the light elements (such as hydrogen and helium) are completely stripped of electrons

both in the pressure ionization region and in the temperature ionization region, and even in the intermediate region between these two regions[3], because of the very high temperatures in stellar interiors. Moreover it is fortunately reliably accepted that in stellar interior the ratio of abundance of heavy elements to that of light elements is only few per cent[4]. Then the number of free electrons per unit atomic weight may be considered to be almostly constant throughout the stellar interiors. Furthermore Coulomb interaction energy between charged particles is negligiblely small compared with the kinetic energy of particles under the conditions of complete ionization. Therefore in stellar interiors, the conditions imposed in the last part of preceding paragraph are sufficiently satisfied, and the relations (13), (14) and (15) safely hold. These conditions pave the way for us to work on the present subject without toil.

Now let xR be the radial distance from the centre of a star, where R is the radius of the star. From the equation (11), we can get the following equation

$$\frac{d \log \Phi}{dx} = \frac{d \log P}{dx} - \frac{5}{2} \frac{d \log T}{dx} \qquad (16).$$

(i) In the non-isothermal region we can, by (16), obtain the following relation

$$\frac{d\psi}{dx} = \frac{\Phi}{\Phi' T} \left(\frac{d \log P}{d \log T} - \frac{5}{2} \right) \frac{dT}{dx} \qquad (17).$$

When the energy is transported by the mode of radiation or conduction, the condition

$$\frac{d \log P}{d \log T} < \left(\frac{d \log P}{d \log T} \right)_{ad} \qquad (18)$$

must be satisfied, because if the condition

$$\frac{d \log P}{d \log T} \geqq \left(\frac{d \log P}{d \log T} \right)_{ad}$$

is satisfied[5], the energy must be mainly transported by the mode of convection.

Since the values of Φ, Φ' and T are all positive, we can, by making use of equations (17), (18) and (13), get the following relation

$$\frac{d\psi}{dx} \Big/ \frac{dT}{dx} > 0$$

and get the final relation

$$\frac{d\psi}{dx} \leqq 0 \qquad (19)$$

bacause in the stellar interior $\frac{dT}{dx} < 0$, where the sign of equality is hold when and only when the transfer of energy is governed by convective process.

(ii) In the isothermal region, we can, by (16), get the following relation

$$\frac{d\psi}{dx} = \frac{\Phi}{\Phi' P} \frac{dP}{dx} < 0 \qquad (20)$$

because the values of Φ, Φ' and P are positive, while $\frac{dP}{dx} < 0$.

Thus we can, by the relations (19) and (20), get the following result:

The degree of electron degeneracy in the stellar interior, decreases outword in the region where the transfer of energy is governed by radiative or conductive process, and is constant in the region where the transfer of energy is governed by convective process.

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ON THE MAGNIFICATION OF THE RECORD OF A VIBRATION BY AN ELECTROMAGNETIC-TYPE TRANSDUCER AND A GALVANOMETER

Ryuzo ADACHI

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The magnification of the record of a vibration $A\sin nt$ by an electromagnetic-type transducer and a galvanometer in the stationary state are

$$\beta_1 = \frac{kn^3}{z_0 z_1} \cdots\cdots\cdots\cdots \text{ for displacement}$$

$$\beta_2 = \frac{kn^2}{z_0 z_1} \cdots\cdots\cdots\cdots \text{ for velocity}$$

$$\beta_3 = \frac{kn}{z_0 z_1} \cdots\cdots\cdots\cdots \text{ for accerelation}$$

where k=*some constant*, $z_0 = \sqrt{(n_0^2 - n^2)^2 + 4\varepsilon_0^2 n^2}$, $z_1 = \sqrt{(n_1^2 - n^2)^2 + 4\varepsilon_1^2 n^2}$.
We discussed how to find the values of damping coefficients ε_0, ε_1 so as to β_1, β_2, β_3 are approximately constant by approaching their two maxima and one minimum in some interval of n for given n_0 and n_1, and get a result which is illustrated by Fig. 3, 4 and 5.

§ 1. General Relations

Let the vibration which will be recorded be

$$x(t) = A\sin nt \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

and let θ, φ be the deflection angles of transducer and galvanometer respectively, then we have

$$\left.\begin{aligned} &\ddot{\theta} + 2\varepsilon_0\dot{\theta} + n_0^2\theta = -K\ddot{x} \\ &\ddot{\varphi} + 2\varepsilon_1\dot{\varphi} + n_1^2\varphi = F\dot{\theta} \end{aligned}\right\} \cdots\cdots\cdots\cdots \quad (2)$$

where ε_0, ε_1, n_0, n_1, K, F are some constants, and we neglect the influence of the electromotive force induced by the motion of the coil of the galvanometer for the transducer. As the initial conditions we put

$$\left.\begin{aligned} &\theta(0) = 0, \qquad \dot{\theta}(0) = -K'\dot{x}(0) \\ &\varphi(0) = 0, \qquad \dot{\varphi}(0) = 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots \quad (3).$$

Then we have following solution

$$\varphi(t)=-FKAn\sum_{i=1}^{4}\frac{\alpha_i^3}{B_i(n^2+\alpha_i^2)}e^{\alpha_i t}+F(K-K')An\sum_{i=1}^{4}\frac{\alpha_i}{B_i}e^{\alpha_i t}$$
$$+\frac{FKA\,n^3}{z_0\,z_1}\sin\left(nt-\tau_0-\tau_1+\frac{\pi}{2}\right) \cdots\cdots\cdots *(4),$$

where

$$\alpha_1+\alpha_2=-2\varepsilon_0,\ \ \alpha_1\alpha_2=n_0^2,\ \ \alpha_3+\alpha_4=-2\varepsilon_1,\ \ \alpha_3\alpha_4=n_1^2 \cdots\cdots\cdots (5)$$

$$\left.\begin{array}{l} z_0=\sqrt{(n_0^2-n^2)^2+4\varepsilon_0^2 n^2},\quad z_1=\sqrt{(n_1^2-n^2)^2+4\varepsilon_1^2 n^2} \\ \tan\tau_0=2\varepsilon_0 n/(n_0^2-n^2),\quad \tan\tau_1=2\varepsilon_1 n/(n_1^2-n^2) \\ B_1=(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4),\quad B_2=(\alpha_2-\alpha_3)(\alpha_2-\alpha_4)(\alpha_2-\alpha_1) \\ B_3=(\alpha_3-\alpha_4)(\alpha_3-\alpha_1)(\alpha_3-\alpha_2),\quad B_4=(\alpha_4-\alpha_1)(\alpha_4-\alpha_2)(\alpha_4-\alpha_3) \end{array}\right\} \cdots\cdots (6).$$

Here ε_0, ε_1 are both real positive, and the real parts of α_1, α_2, α_3, α_4 are always negative no matter $\alpha_i's$ are real or imaginary. Therefore if t is large $|e^{\alpha i t}|$ is very small and we get following approximate formula

$$\varphi(t)=\frac{FKA\,n^3}{z_0\,z_1}\sin\left(nt-\tau_0-\tau_1+\frac{\pi}{2}\right) \cdots\cdots\cdots (7).$$

The displacement, the velocity and the accerelation of the original vibration are

$$A\sin nt,\quad An\sin\left(nt+\frac{\pi}{2}\right),\quad An^2\sin(nt+\pi)$$

respectively, and the magnifications of the record $\varphi(t)$ concerned above are

$$\beta_1=\frac{FK\,n^3}{z_0\,z_1},\quad \beta_2=\frac{FK\,n^2}{z_0\,z_1},\quad \beta_3=\frac{FK\,n}{z_0\,z_1} \cdots\cdots\cdots (8).$$

§ 2. Magnification for Displacement

Putting

$$\varepsilon_0/n_0=h_0,\quad \varepsilon_1/n_1=h_1,\quad 2h_0^2-1=\xi_0,\quad 2h_1^2-1=\xi_1 \cdots\cdots\cdots (9)$$

gives

$$-1<\xi_0,\quad -1<\xi_1$$

$$z_0=\sqrt{n^4+n_0^4+2\xi_0 n_0^2 n^2},\quad z_1=\sqrt{n^4+n_1^4+2\xi_1 n_1^2 n^2} \cdots\cdots\cdots (10)$$

and

$$\frac{d}{dn}\beta_1(n)=\frac{-FK\,n^2}{z_0^3\,z_1^3}F_1(x) \cdots\cdots\cdots (11)$$

where

$$\left.\begin{array}{l} L_1=n_0^4+n_1^4+4\xi_0\xi_1 n_0^2 n_1^2 \\ M_1=-4n_0^2 n_1^2(\xi_0 n_1^2+\xi_1 n_0^2) \\ N_1=3n_0^4 n_1^4 \\ x=n^2 \end{array}\right\} \cdots\cdots\cdots (12)$$

* See references (3)

$$F_1(x) = x^4 - L_1 x^2 + M_1 x - N_1 \quad \cdots\cdots\cdots\cdots\cdots (13)$$

Therefore the values of n which give minimum or maximum to $\beta_1(n)$ are the square roots of the positive roots of $F_1(x)=0$.

It is easily proved that the equation $F_1(x)=0$ has at least one positive root, and generally it can have three positive roots. When $F_1(x)=0$ has only one positive root, $\beta_1(n)$ has only one maximum value and as shown in Fig. 1, we can not admit that $\beta_1(n)$ is nearly constant.

Next, the condition that $F_1(x)=0$ has three positive roots is

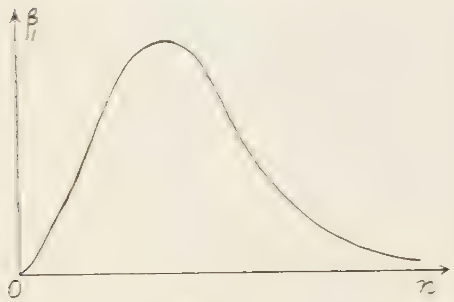

Fig. 1

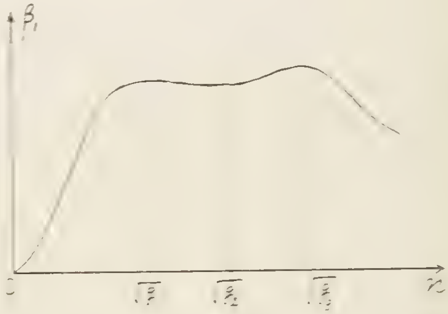

Fig. 2

$$\left.\begin{array}{l} L_1>0, \quad M_1>0, \quad 9M_1^2-32L_1N_1>0 \\ \qquad 9M_1^2-8L_1N_1-2L_1^3<0 \\ (27M_1^2-72L_1N_1-2L_1^3)^2M_1^2-(9M_1^2-8L_1N_1-2L_1^3)^2(9M_1^2-32L_1N_1)<0 \end{array}\right\} \cdots\cdots (14)$$

and in this case, let the positive roots of $F_1(x)=0$ be g_1, g_2, g_3 [illegible], then $\beta_1(n)$ takes maxima at $n=\sqrt{g_1}, \sqrt{g_3}$, minimum at $n=\sqrt{g_2}$. If

$$[illegible]$$

are both small and $(\sqrt{g_3}-\sqrt{g_1})$ is large, we can recognize that $\beta_1(n)$ is nearly constant when $\sqrt{g_1} \leqq n \leqq \sqrt{g_3}$ as shown in Fig. 2.

If we use electromagnetic oscillograph, n_2/n_1 is very large, and putting

$$n_2 \gg n_1, \quad \xi_1 = 0(1), \quad \xi_2 \leqq 0\left(\frac{n_1}{n_2}\right)$$

gives

$$L_1 [illegible], \quad M_1 [illegible], \quad N_1 = [illegible]$$

and from (14) we get

$$[illegible]$$

hence

$$-1<\xi_0<-\frac{\sqrt{3}}{2} \quad \cdots\cdots\cdots (15)$$

$$0<h_0<\frac{\sqrt{6}-\sqrt{2}}{4} \quad \cdots\cdots\cdots (16)$$

and

$$F_1(x)\equiv x^4-n_1^4x^2-4n_1^4n_0^2\xi_0x-3n_0^4n_1^4=0 \quad \cdots\cdots\cdots (17).$$

From this equation we get

$$\left.\begin{array}{l} g_1\doteqdot(-2\xi_0-\sqrt{4\xi_0^2-3})n_0^2 \\ g_2\doteqdot(-2\xi_0+\sqrt{4\xi_0^2-3})n_0^2 \\ g_3\doteqdot n_1^2 \end{array}\right\} \quad \cdots\cdots\cdots (18)$$

and

$$\left.\begin{array}{l} \beta_1(\sqrt{g_1})\doteqdot\dfrac{FKn_0}{n_1^2}\dfrac{\{-2\xi_0-\sqrt{4\xi_0^2-3}\}^{3/2}}{\sqrt{4\xi_0^2-2+2\xi_0\sqrt{4\xi_0^2-3}}} \\ \beta_1(\sqrt{g_2})\doteqdot\dfrac{FKn_0}{n_1^2}\dfrac{\{-2\xi_0+\sqrt{4\xi_0^2-3}\}^{3/2}}{\sqrt{4\xi_0^2-2-2\xi_0\sqrt{4\xi_0^2-3}}} \\ \beta_1(\sqrt{g_3})\doteqdot\dfrac{FK}{2\varepsilon_1} \\ \sqrt{g_3}-\sqrt{g_2}\doteqdot n_1 \end{array}\right\} \quad \cdots\cdots\cdots (19).$$

Accordingly, at first determine the value of ε_0 so as to $\beta_1(\sqrt{g_1})\doteqdot\beta_1(\sqrt{g_2})$ and next ε_1 so as to $\beta_1(\sqrt{g_2})\doteqdot\beta_1(\sqrt{g_3})$, then we can recognize that $\beta_1(n)$ is nearly constant in the interval $\sqrt{g_1}\leq n\leq n_1$.

Solving the equation $\beta_1(\sqrt{g_1})=\beta_1(\sqrt{g_2})$ with respect to ξ_0, we get only one root $\xi_0=-\sqrt{3}/2$, and in this case

$$\beta_1(\sqrt{g_2})\doteqdot 3^{3/4}FKn_0/n_1^2,\quad h_0\doteqdot 0.259 \quad \cdots\cdots\cdots (20).$$

Putting $\beta_1(\sqrt{g_3})=\beta_1(\sqrt{g_2})$ gives*

$$h_1=\frac{\varepsilon_1}{n_1}\doteqdot\frac{1}{4.56}\times\frac{n_1}{n_0} \quad \cdots\cdots\cdots (21)$$

For example, when

$$n_0=2\pi\times 3/sec,\quad n_1=2\pi\times 100/sec$$

putting

$$h_0\doteqdot 0.26,\quad h_1\doteqdot\frac{1}{4.56}\times\frac{100}{3}\doteqdot 7.3$$

* $\xi_0=-\sqrt{3}/2$ and (15) or $\beta_1(\sqrt{g_2})<\beta_1(\sqrt{g_3})$ and $\beta_1(\sqrt{g_2})=\beta_1(\sqrt{g_3})$ can not stand together, but above calculation is rough approximation, and we only know that ξ_0 is near $-\sqrt{3}/2$. Hence at any rate, test the value of $\beta_1(n)$ by putting $\xi_0=-\sqrt{3}/2$ and if it is not suitable for our purpose, amend this value by successive approximation.

we get

$$\xi_0 \fallingdotseq -0.87, \quad \xi_1 \fallingdotseq 106$$

and

$$\beta_1(p) = \frac{FK}{2\pi} \frac{p^3}{\sqrt{p^4 - 15.6\,p^2 + 81}\,\sqrt{p^4 + 212\times 10^4\,p^2 + 10^8}}\,sec \cdots\cdots\cdots \quad (22)$$

where $n = 2\pi p$ and p is the number of frequency per second. By the numerical calculation of (22) we get Fig. 3 and this curve shows that above values of h_0, h_1 are not suitable enough so as to $\beta_1 \fallingdotseq const.$ Fig. 3, B, C show the curves of $\beta_1(p)$ when $h_0 = 0.26$, $h_1 = 8.6$; $h_0 = 0.26$, $h_1 = 10.0$ respectively. It shows that the error of $\beta_1 \fallingdotseq const$ in B is within

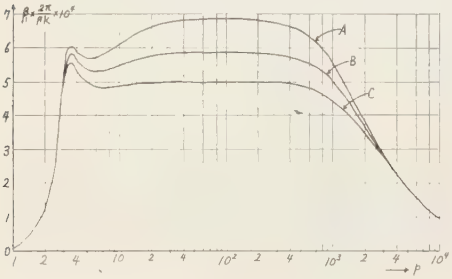

Fig. 3

$$10\%\cdots\cdots\text{when} \quad p = 3\sim 900\,(/sec)$$

$$1\%\cdots\cdots\text{when} \quad p = 30\sim 300\,(/sec)$$

§ 3. Magnification for Velocity

Similarly in above section, we can obtain following result.

When $n_1 \gg n_0$, $\xi_0 = 0(1)$, $\xi_1 = 0(1)$, $\xi_0\xi_1 \neq 0$, β_2 has two maxima at $n = \sqrt{g_1}$, $\sqrt{g_3}$ and one minimum at $n = \sqrt{g_2}$ where

$$g_1 \fallingdotseq -\frac{n_0^2}{\xi_0}, \quad g_2 \fallingdotseq \sqrt{\frac{\xi_0}{\xi_1}}\,n_0 n_1, \quad g_3 \fallingdotseq -\xi_1 n_1^2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (23)$$

and we have

$$\left.\begin{array}{l} \beta_2(\sqrt{g_1}) \fallingdotseq \dfrac{FK}{n_1^2\sqrt{1-\xi_0^2}}, \quad \beta_2(\sqrt{g_2}) \fallingdotseq \dfrac{FK}{n_1^2} \\ \beta_2(\sqrt{g_3}) \fallingdotseq \dfrac{FK}{n_1^2\sqrt{1-\xi_1^2}} \\ \sqrt{g_3} - \sqrt{g_1} = \sqrt{-\xi_1}\,n_1 - \sqrt{-\dfrac{1}{\xi_0}}\,n_0 \end{array}\right\} \cdots\cdots\cdots\cdots\cdots\cdots \quad (24).$$

From $\beta_2(\sqrt{g_1}) = \beta_2(\sqrt{g_3})$ we get $\xi_0 = \xi_1$ and $\beta_2(\sqrt{g_1}) - \beta_2(\sqrt{g_2})$ is increasing function of $|\xi_0|$.

For example, putting

$$\xi_0 = \xi_1 = -0.25, \quad n_0 = 2\pi \times 3/sec, \quad n_1 = 2\pi \times 100/sec$$

gives

$$g_1 \fallingdotseq (2\pi \times 6)^2/sec^2, \quad g_2 \fallingdotseq (2\pi \times 17)^2/sec^2$$

$$g_3 \fallingdotseq (2\pi \times 50)^2/sec^2$$

and

$$\beta_2(\sqrt{g_1}) \doteqdot \beta_2(\sqrt{g_3}) \doteqdot \frac{1.03\,FK}{n_1^2}\ sec^2$$

$$\beta_2(\sqrt{g_2}) \doteqdot \frac{FK}{n_1^2}\ sec^2.$$

Therefore the error of $\beta_2 = const$ may be within 3 % when $p = 6 \sim 50$. The actual numerical calculation by (8) is shown in Fig. 4 and it indicates that the error of $\beta_2 = const$ is within 2 % when $p = 4.4 \sim 68$ and above conjecture is correct.

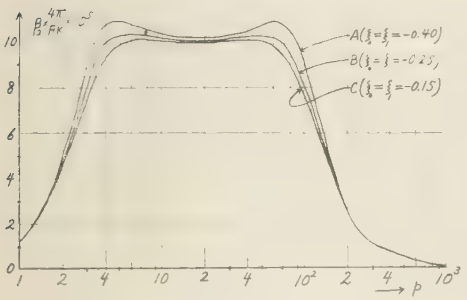

Fig. 4

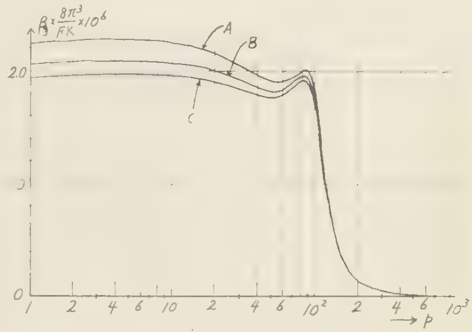

Fig. 5 A…$h_1 = 0.26$, $h_0 = 7.3$
B…$h_1 = 0.26$, $h_0 = 8.0$
C…$h_1 = 0.26$, $h_0 = 8.5$

§ 4. Magnification for Accerelation

Quite similar treatment, we get Fig. 5 when $n_0 = 2\pi \times 3/sec$, $n_1 = 2\pi \times 100/sec$.

§ 5. Conclusion

As well known, we can obtain good record by using suitable transducer and galvanometer according to the frequency p of a vibration. But, in this case, we must prepare many kind of transducer and galvanometer for various values of p and in general, the sensitivity is very small. Therefore a good amplifier is wanted. This is a great expense.

On the contrary, the argument of this paper is a result of that to find a interval of p in which the magnification of the record of a vibration is approximately constant by a pair of a transducer and a galvanometer which are in our hand not largely lowering the sensitivity by putting the damping coefficients suitablly.

When we use oil immersion damping or air damping, to give a given value to ε (in other words h) is very difficult, but when we use electromagnetic transducer and galvanometer, it is easy by connecting suitable electric resistance in the circuit.

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THE INVESTIGATION OF "SARA-ISI" (Report 4) THE CEMENTING MATERIALS IN THE MANTLE PART OF "SARA-ISI"

Tadashi OKAHATA

(Received February 1, 1959)

Introduction

Volcanic ashes of Mt. Aso are ordinarily called "Yona". Particles of Yona are broadly divisible into three classes, large, medium and small particles, under the microscope. In generally, some of these medium and small particles stick around to the large ones to form many clusters.

Writer made a suspension by stirring Yona in the water, and by using the depositing velocity of particles in it, divided them into four classes. (Table 1)

Table 1.

1. large particles (the rapidly deposited):	0.1 ~0.15 mm
2. comparatively large ones among medium particles:	0.04 ~0.09 mm
3. comparatively small ones among medium particles:	0.02 ~0.04 mm
4. minute particles:	0.002~0.004 mm

Between the medium particles 3 and minute ones 4 can imagine some particles which are $0.004 < x < 0.02$ mm in size, but they are not recognized practically. The reason why remain unexplained. It may possibly be characteristic of all volcanic ashes.

To investigate the soluble ingredients in water, another suspension of Yona was made, and left it alone for a day, and after that, the filtrate was taken from it. This was a colorless, transparent and weak acid liquid. From this liquid, after the gradual drying, minute and white crystals precipitated.

On the other hand, the mantle part scrapted off from Sara-isi was powdered in the mortar and put into the water. After leaving it for a day, the filtrate was gained. This was a colorless, transparent and weak acid solution, too, and from this solution the minute white crystals were gained similarily as the above stated about the case of Yona. It was ascertained that the acidity of these filtrates was due to the existence of free sulphuric acid.

The results of the investigation with the groups of particles in the table 1, and with the white crystals are as follows.

Microscopic Examination to the Groups of the Particles of Yona.

Similarly as the results of the microscopy to the specimens from the mantle part

of Sara-isi[1], in every group there seemed the crystals of plagioclase, magnetite and augite, but quartz not at all. In the group of smaller particles, the amount of these crystalline ingredients grew less and on the other hand, the amorphous ones grew larger, and especially in the minute group, the light was not permited under the cross nicol, viz. the quantity of the amorphous ingredients was very large. But, in this case, there are few crystals of these kind distinguishable in the size about 1/1000 mm.

Examination to the Groups of the Particles by the X-ray.

Fig. 1, 2, 3 and 4 in the plate I are the diffraction patterns of 1, 2, 3 and 4 in the table 1, respectively. Particles of the group 1 in table 1 were kneaded with grease into a thin plate 0.2 mm and affixed on the slit directly and was exposed.

Every specimen corresponding to the fig. 2, 3, and 4 have been prepared by pressing these particles to a thin plate respectively, and they were affixed directly on the slit, at every time.

Fig. 1 shows a random aggregation of diffraction spots, and this due to the presence of pretty large crystal grains in this specimen.

Fig. 2 and 3 show a peppering of diffraction spots and continuous Debye–Scherrer rings. And, on the other hand, Fig. 4 shows the continuous Debye–Scherrer rings only.

By the way, Fig. 5 is the diffraction patterns for the specimen which has been prepared from the mantle part of Sara–isi, and Fig. 6 is the one for the specimen from the stalactitic processes of it.

The pattern of Fig. 5 have many points of likeness to the one of Fig. 2 or Fig. 3, and Fig. 6 to Fig. 4.

Examination to the Crystal gained from the Filtrate.

Both crystals gained from the filtrates which were made from Yona and the powder of mantle part of Sara-isi, bear some resemblance in the shape and cleavages to the crystal of gypsum.

According to the chemical analysis by T. Asahina and Y. Miyake[2][3], when Yona 5 gr was put into the water 50 cc, the solution's pH. was 3.6, dissolved part was 0.255 gr, and its percentage in Yona was 5.1 %. And the chemical composition of this dissolved part was as follows.

Table 2.

Al	5.9 (wt.%)
Ca	9.8
Mg	6.6
Mn	0.3
SO_4	61.4
Cl	15.3
H_2O and others	0.7
Totale	100.0

The presence of gypsum would be imaginable by considering above results. By the way, the solubility of gypsum in water is 0.23 gr at 0°C and 0.22 gr at 100°C.

Writer examined the refractive index of this crystal by the immersion method of microscope, and gained confidence that they were gypsum.

Then lastly, powder photographs of these crystals were taken and compared with each other and with the one of gypsum. The results are shown in the Fig.

7 and Fig. 8 of plate II.

The left of the Fig. 7 of plate II is the powder photograph of the crystal educed from the solution of the mantle part of Sara-isi, and right is the one of crystal of gypsum. In Fig. 8, the left is the same as the right of Fig. 7, and right is the powder photograph of the crystal educed from the solution of Yona. Between the left and right in these figures, Debye-Scherrer rings make only little difference with one another in the sharpness of the rings, but position of them wholly coincide with each other. Hereupon, the writer can conclude that the crystalline substance are gypsum.

Chemical Composition of the Mentle Part of Sara-Isi

At writer's request, the analysis of the mantle part of Sara-isi were kindly performed by Mr. M. Hayakawa, a laboratory man of Geological Survey Institute, to whom the writer expresses his best thanks here. The results is shown in table 3.

Table 3

chemical composition of mantle part of Sara-isi.

	(wt. %)
SiO_2	57.06
Al_2O_3	11.73
Fe_2O_3	5.92
FeO	3.52
MnO	0.09
MgO	2.10
CaO	4.49
Na_2O	1.75
K_2O	1.48
TiO_2	0.13
P_2O_5	0.75
H_2O^+	5.76
H_2O^-	2.78
SO_3	2.28
Total	99.84

Table 4

Norm (a)			Norm (b)	
q.	(quartz)	30.54	q.	27.72
or.	(orthoclase $K_2O.Al_2O_3.6SiO_2$)	8.90	or.	8.90
ab.	(albite $Na_2O.Al_2O_3.6SiO_2$)	15.21	ab.	14.67
an.	(anorthite $CaO.Al_2O_3.2SiO_2$)	16.13	an.	19.74
wo.	(wollastonite $CaO.SiO_2$)	0	wo	0.81
en.	(enstatite $MgO.SiO_2$)	5.3	en.	5.30
fs.	(ferrosilite $FeO.SiO_2$)	0.4	fs.	3.04
anhy.	(anhydrite $CaSO_4$)	3.95	ap.	0.67
ap.	(apatite $(CaO)_3.P_2O_5$)	0.67	il.	1.52
il.	(ilmenite $FeO.TiO_2$)	1.52	mg.	6.26
mg.	(magnetite $FeO.Fe_2O_3$)	8.58	sulphur	0.93

Table 4 is Norm which was calculated by the writer from the results indicated in the table 3. First column (a) in this table shows Norm which was calculated from regarding SO_3 in table 3 as gypsum ($CaO.\ SO_3$), and second column (b), exhibited only by way of information, is the one regarding SO_3 as the free sulphur S. The quantity of quartz (glass state, in practice) are conspicuously great (about 30%) in comparison with the others, either in (a) or in (b).

Next, expressed the three-components diagram of composition of felspar and pyroxene indicated in the column (a) and (b), with the aid of trilinear coordinates. (Fig. 9 and 10). And, on the an-ab line in Fig. 9 dotted the composition of plagioclase.

Point (a') and (b') corresponds to the composition of plagioclase indicated in the column (a) and (b), respectively. Both point represents the laboradorite belongs to the acid member.

Conclusion

From the preceding examinations and studyings with the results of chemical analysis, may conclude as follows.

The mineral crystal elements in the mantle part of Sara-isi and in Yona are the same in their kinds. Components of Norm expression which originated in the result of chemical analysis with the mantle part of Sara-isi, give a good support to the above mentioned. Therefore, the marginal part of Sara-isi is the coagurated part of the fine particles of the volcanic ashes, Yona, smaller than about 0. 09 mm of particles at that; i. e. writer's the so-called "medium particles" and "minute particles".

Hitherto, writer have been thought solely the colloidal silicic anhydride as the coagurating (cementing) material at the mantle part of Sara-isi. As a matter of fact, from the examinations by the mineral microscope or x-ray, not-with-standing there can not find the crystalline quartz in the specimen which was cut from the mantle part of Sara-isi, but according to the results of chemical analysis, the quantity of the free silicic anhydride comes up to 30 % in Norm, as is shown in table 4; viz. after eliminating the element SiO_2 containing in the composition of felspar and pyroxene from its total quantity, still remain 30 %, in the end. Therefore, it may safely be said that the colloidal silicic acid are the cementing material in the mantle part. Besides, about the possibility of the production of colloidal silicic acid in the Aso-crater, Dr. M. Namba reported in his paper, already[(4)].

In conclusion, writer found out gypsum from the filtrate which was made from the powder of the mantle part of Sara-isi. It is true that its measure (3~4%) is far little compared with the colloidal silicic acid, but it may safely be said that is also one of a cementing material.

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(2) T. Asahina and Y. Miyake; J. Meteo. Soc. 2nd Series, Vol. XIII, No. 12 (1935)
(3) Y. Miyake; J. Meteo. Soc. 2.nd Series, Vol. XVI, No. 3 (1938)
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Plate I

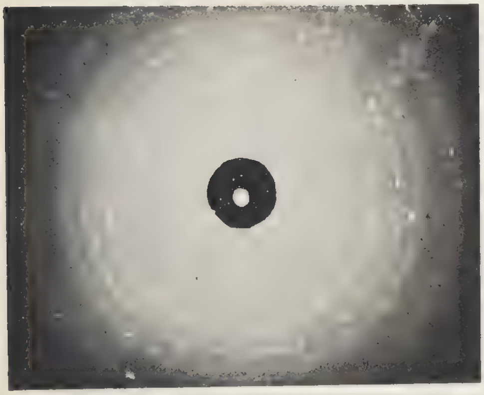


Fig. 3

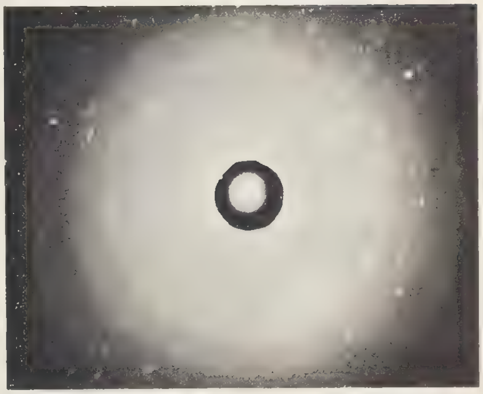


Fig. 6

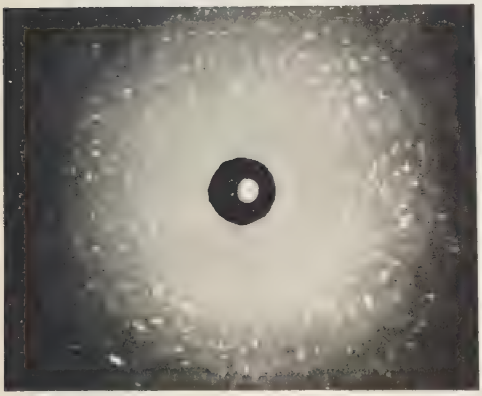


Fig. 2

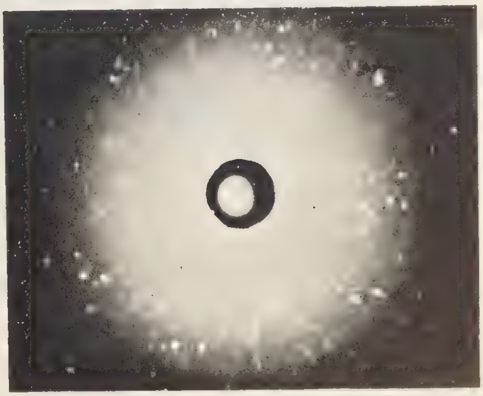


Fig. 5

Fig. 1

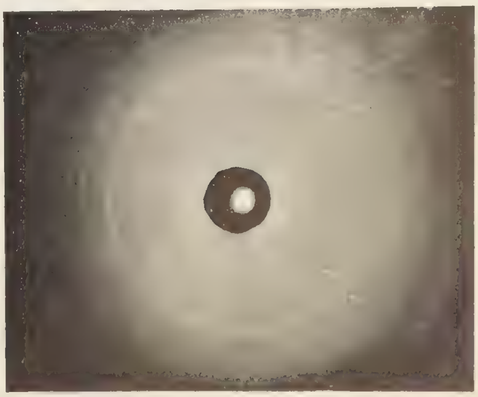


Fig. 4

Plate II

Powder photographs

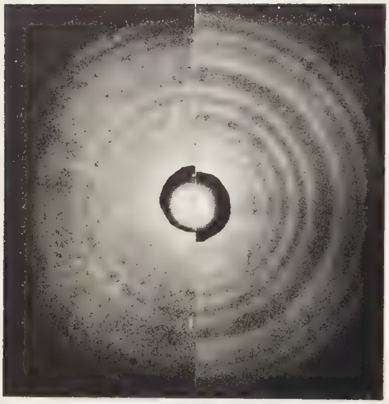


Fig. 7

left: crystal obtained from the mantle part of Sara-isi.
right: gypsum

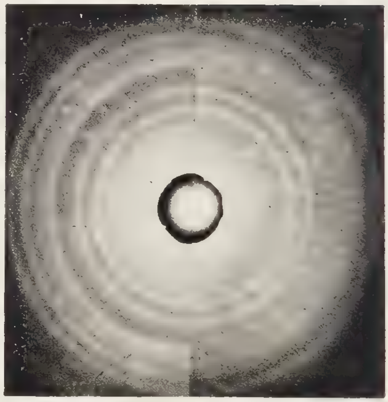


Fig. 8

left: gypsum
right: crystal obtained from the solution of Yona.

Plate III

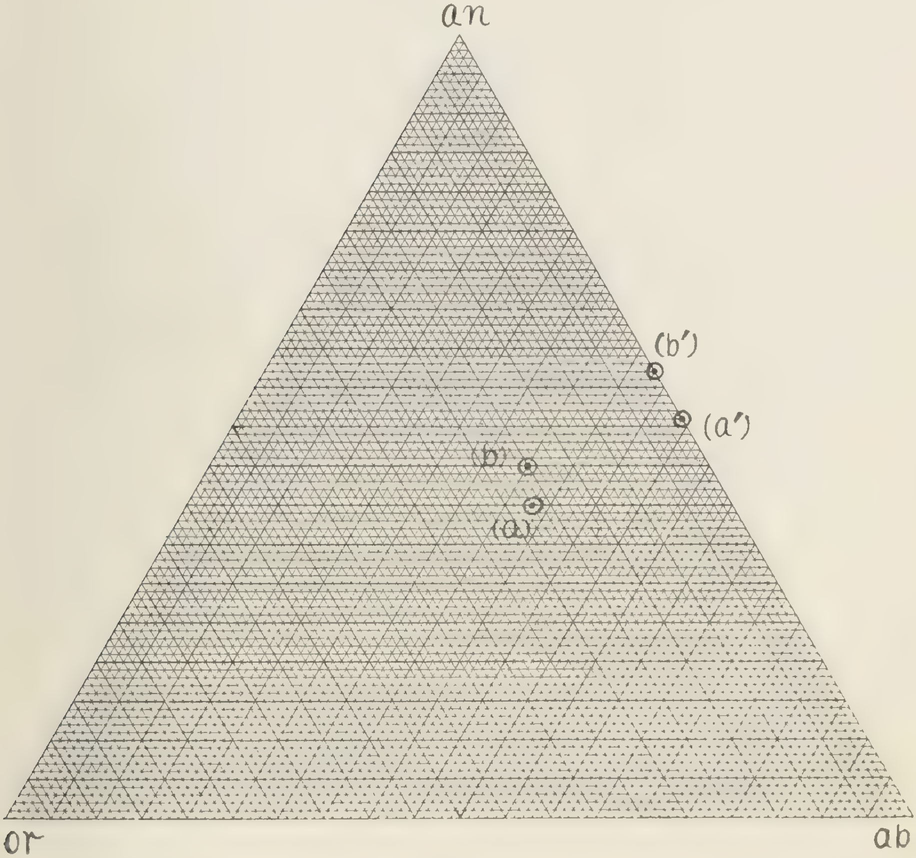


Fig. 9

Felspar and Plagioclase

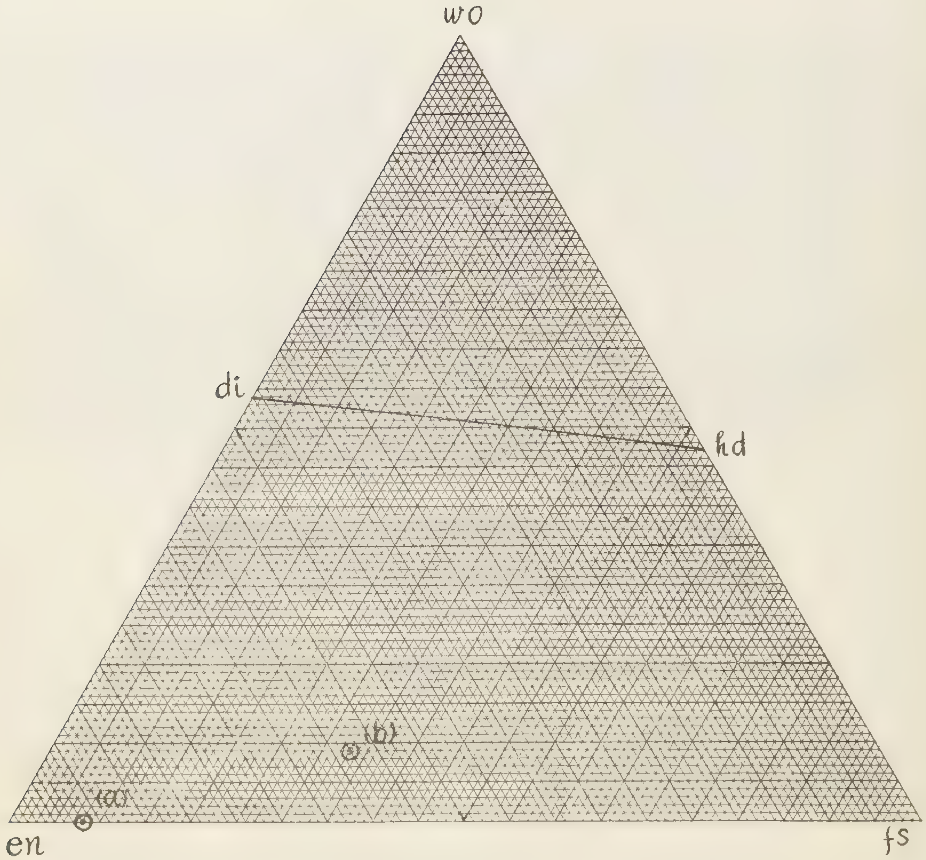


Fig. 10
Pyroxene

A HYDROLOGICAL STUDY ON THE OLD CRATER OF ASO

Tosisato MUROTA

(Received November 1, 1958)

1. Introduction

The old crater of Aso is considered to be the greatest depression caldera in the world. The shape is nearly an ellipse whose major axis is about 25 km from south to north and minor axis is about 19 km from east to west. The circumference is about 80 km and the area is about 370 km². New volcanoes erupted in the central zone of this old crater field and form the central volcanic cones to which active volcano

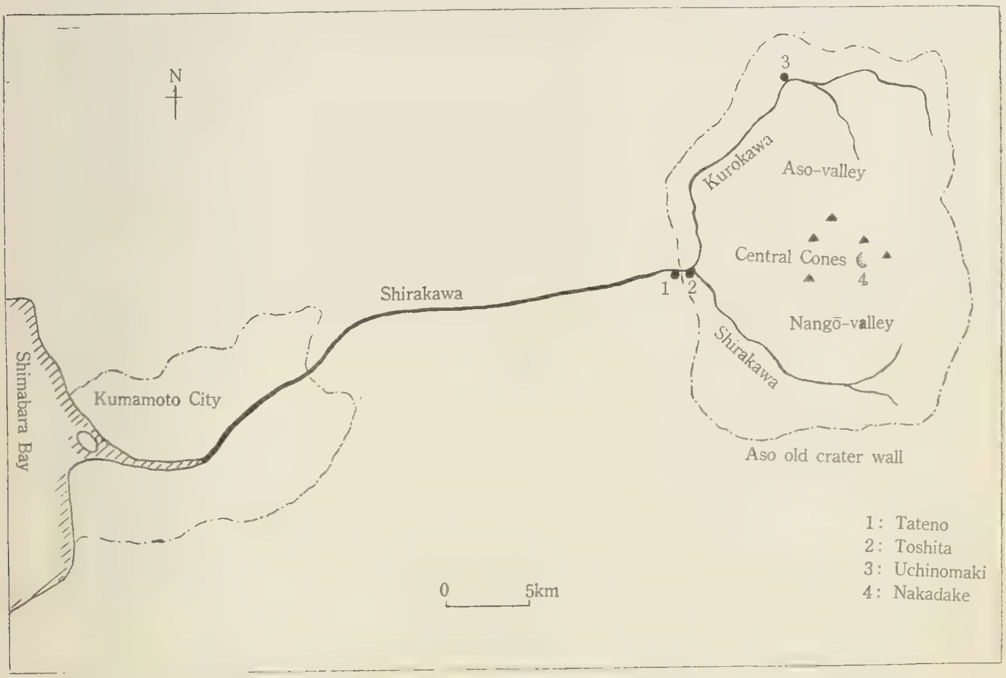


Fig. 1 The Shirakawa

Nakadake belongs. These central cones seperate the old crater into two valleys north and south. The north valley is called Aso-valley and the south one Nango-valley.

The Kurokawa, running inside the Aso-valley (catchment area about 195 km²), and the Shirakawa, running inside the Nango-valley, meet each other at Toshita which stands at the west border of Aso crater atrio, and after their confluence, the trunk

river flows through Tateno-ravine of crater wall and, running through the Plain of Kumamoto from east to west, it finally flows into Shimabara-bay. This river system whose source lies in Aso old crater is called the Shirakawa.

The geological features of Aso old crater consist of layers of volcanic ashes and gravel among which lava and clay are mixed. Therefore its characteristic feature is that the infiltration capacity is large. This is one of the causes of the variations of the water-level of river and the head of hot spring, about which the writer has reported in the previous papers,[1][2] and it must strongly influence upon the discharge due to rainfall over Aso old crater.

In general, infiltration is an important factor to determine the relations between precipitation and runoff. But it is so difficult to treat infiltration that there have been few examples treating it satisfactorily.

Aso region, where the infiltration capacity is large, is considered to be an appropriate region to research the effect of the infiltration on the discharge due to precipitation. On this point of view the writer tried to analyse, taking the infiltration into account, the river discharge due to rainfall over Aso old crater.

The data used are the records of stage of the Kurokawa at Uchinomaki Observation Station of Volcano and Hot-Spring Laboratory of Kyoto University which stands at the mid-stream of the Kurokawa. The data were recorded by a self-recording gauge in natural scale in order to record minute variations of the water-level. The amount of precipitation R mm/hour and the water-level H cm at every hour are given partly in the Kumamoto Journal of Science Vol. 3, No. 2, 1957 and partly at the end of this paper. The data of precipitation are those recorded by the self-recording gauge of Uchinomaki Observation Station. Since there were only a few pluviometric posts in the drainage basin of the Kurokawa at that date, it is difficult to estimate exactly the amount of rain which falls over the drainage basin of the Kurokawa up stream Uchinomaki. However, since the drainage area of the Kurokawa is not very wide, we take the above mentioned records for the representatives of the amount of rain. But we have excluded the records of precipitation which are considered to be inadequate as representatives of the amount of the precipitation of drainage basin, judging from the comparison of those with the data of hourly precipitation of Aso Cold District Experimental Station as well as the data of daily precipitation of Mt. Aso Weather Station. Therefore, the data used by the writer may be regarded as adequate.

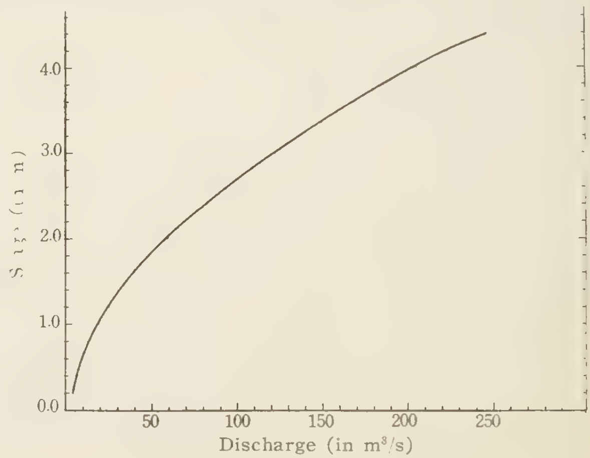


Fig. 3 Discharge-Stage Curve

Fig. 2 Aso District

In order to analyse the discharge we need the amount of discharge instead of the water-level. Kumamoto Construction Office of Construction Department appointed as a discharge gauging station the point where the river gauge of Uchinomaki Observation Station was set, and obtained the discharge-stage curve, which we use in the present paper.

2. Theoritical Consideration on the Analysis of Discharge due to Rainfall, taking Infiltration into Account

The relation between rainfall and discharge is the key problem of hydrology and has been studied by researchers. As for the method by which the discharge is computed from the rainfall, the method due originally to J. A. Folse and named "Unit-graph method" by L. K. Sherman[3] has been used widely and has been improved by many investigators. It seems that this method gives fairly good results for rivers which have moderate drainage basins.

The so-called unitgraph method is an emperical law and has no clear theoretical foundation, so that its applicability is judged only by experience. R. H. Horton tried to calculate the discharge taking the equation of motion and infiltration into account. His idea is excellent, but his method called "Infiltration method" is unsatisfactory and is not practical since he made several assumptions. S. Hayami, developing this method, discussed a general theory of deriving discharge from rainfall and laid a foundation for the unitgraph method. In principle, the unitgraph method must be applied to surface runoff, ground water discharge and other phenomena, each of which is governed by a certain characteristic law respectively. In this method it is assumed that the relation between rainfall and discharge is linear. Exactly speaking, however, the actual relation between rainfall and surface runoff is non-linear, and the form of runoff is different for every rainfall. The experiential fact that the unitgraph method is practically of use means that the relation between rainfall and surface runoff can be expressed approximately by a linear relation.

Assuming, then, that the relation between the intensity of rainfall R and the surface runoff Q is linear, S. Hayami has given the following equation;

$$Q(t)=\int_0^t G(t-\tau)[R(\tau)-I(\tau)]d\tau \quad \cdots\cdots \quad (1)$$

where I is the rate of infiltration and G is a function which expresses the operation that transforms rainfall into surface runoff and is a function of time t for a fixed observation point. These three quantities R, I and Q are expressed by the values per unit time divided by the drainage area, i. e. in depth of water. For uniform rain of one hour duration, taking one hour as unit of time, the surface runoff is given by

$$Q(t)=\int_0^t GE\delta(\tau)d\tau-\int_1^t GId\tau, \qquad t>1 \quad \cdots\cdots \quad (2)$$

or

$$\frac{Q}{E}=\int_0^t G\delta(\tau)d\tau-\int_1^t \frac{GI}{E}d\tau, \qquad t>1 \quad \cdots\cdots \quad (2')$$

where

$$E = R - f - I,$$

$$\delta(t) = \begin{cases} 1 \text{ for } & 0 < t \leqq 1 \\ 0 \text{ for } & t > 1 \text{ and } t \leqq 0, \end{cases}$$

f being the loss of rainfall before the beginning of runoff. We call the term $\int_0^t G\delta d\tau$ unitgraph. In this case, the duration of rainfall R is one hour but infiltration I is considered to remain after rainfall is over. If infiltration ceases with rain-stop, we get instead of (2) and (2′)

$$Q = \int_0^t GE\delta(\tau)d\tau, \quad \cdots\cdots (3)$$

or

$$\frac{Q}{E} = \int_0^t G\delta(\tau)d\tau. \quad \cdots\cdots (3')$$

Usually, it is assumed that the unitgraph is expressed by (3′), taken one hour as unit of time. In general, however, it is reasonable to consider that infiltration continues as long as surface detention does, even when rainfall ceases. Especially in the region where the infiltration capacity is large as in the case of Aso old crater, the infiltration after the cessation of rainfall can not be neglected and it is not appropriate to express unitgraph by (3′). On the basis of the physical meaning of unitgraph, it is proper to derive unitgraph using (2′) from the observed hydrograph for uniform one hour rain; we must add the second term on the right hand side of (2′) to the observed hydrograph in order to obtain unitgraph. The method is described in section 4.

3. Derivation of Ground Flow Hydrograph

The observed flood discharge is the sum of the surface runoff and the ground water discharge, so that we must separate it into the surface runoff and the ground water discharge. Ground flow consists of two parts; one (called base flow) is mostly originated in the previous rainfall and the other results from the present rainfall. In general, rainfall affects the ground flow in two ways. One follows immediately after the rainfall, which is considered to be the discharge of ground water in shallow layers as well as the so-called sub-surface flow, and the other is the discharge which continues for a long period after the rain. The former produces a powerful effect on floods and the latter has the same rate of decrement with the flow before the rain and for a rain of short duration, it does not sensibly appear in many cases. Therefore, we treat the latter as being included in the base flow and we call the former the ground water discharge. The base flow can be derived from the discharge before the rain, taking the natural decrease into consideration. Thus the question is to derive the ground water discharge due to the present rain. Hereafter the base flow is excluded from our treatment.

For one hour rain (including two hours rain on record which is actually one hour rain. E. g., the rainfall from $9^h\ 40^m$ to $10^h\ 40^m$ is recorded as two hours rain from

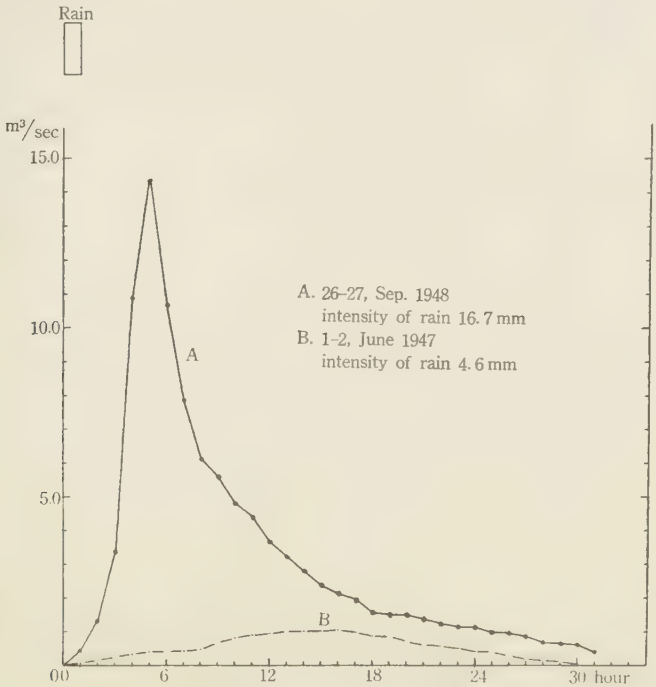
9^h to 11^h) the variation of the discharge at Uchinomaki was examined. For weak rain there are two cases: in one case the discharge does not change practically, while in the other case the ground water discharge is observed, but not surface runoff is observed. Classifying rainfalls according to their intensity, the average increase of discharge due to rainfall is as shown in Table 1.

Table 1. Intensity of rain and etc.

Range of intensity of rain (mm/h)	Average of intensity of rain R mm/h	Increase of discharge Q in mm	$R-Q$ in mm
$0<R<5$	2.75	0.52	2.23
$5<R<15$	10.4	0.61	9.79
$15<R$	24.2	5.46	18.74

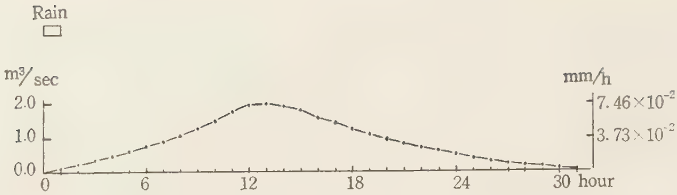
The forms of hydrograph for heavy rain and weak rain, the duration of each of which is one hour, are shown in Figs. 4A and 4B respectively. It is concluded from the arrival time of the peaks that the former contains both of surface runoff and ground water discharge, and the latter consists of ground water flow only. Averaging examples of the latter type and taking one hour as unit of time, we obtain the ground flow hydrograph that corresponds to 1 mm of total discharge. It is shown in Fig. 5 and Table 2.

It is considered that Fig. 5 shows the form of ground water discharge for one hour rain. Since this hydrograph indicates stationary state for several hours at the peak, we can say that the ground water discharge from the rainfall which continues



Time from the beginning of rain

Fig. 4 Hydrograph for one hour rain



Time from the beginning of rain

Fig. 5

more than one hour also has this form generally. Then, we may obtain the surface runoff by subtracting the ground water discharge from the flood discharge (observed flow minus base flow) in the following way. Though the discharge includes surface runoff, the descending part of the hydrograph will have the same rate of decrement as that of Fig. 5, after the surface runoff terminates, namely when only the ground water discharge survives. Therefore the time when the accordance between the rate of decrement of the observed hydrograph and that of Fig. 5 begins is to be the time

Table 2. Ground flow hydrograph

Time from the beginning of rain (hour)	Discharge $(m^3/sec) \times 10^{-2}$
1	10
2	22
3	33
4	46
5	57
6	75
7	84
8	105
9	128
10	149
11	175
12	195
13	196
14	190
15	181
16	155
17	144
18	125
19	110
20	95
21	82
22	71
23	60
24	49
25	37
26	30
27	21
28	18
29	14
30	8
31	6

1 m^3/sec is equivalent to 3.73×10^{-2} mm/hour.

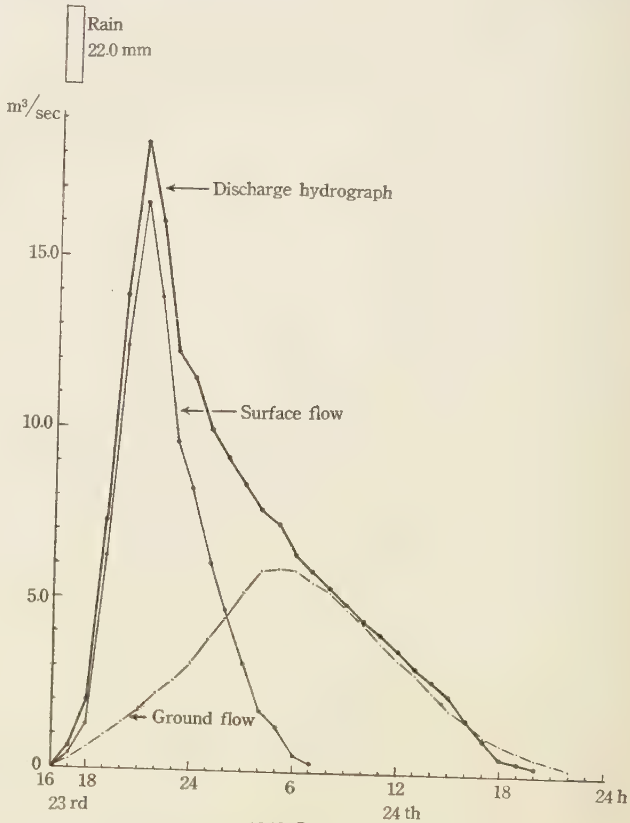


Fig. 6A Hydrograph for one hour rain

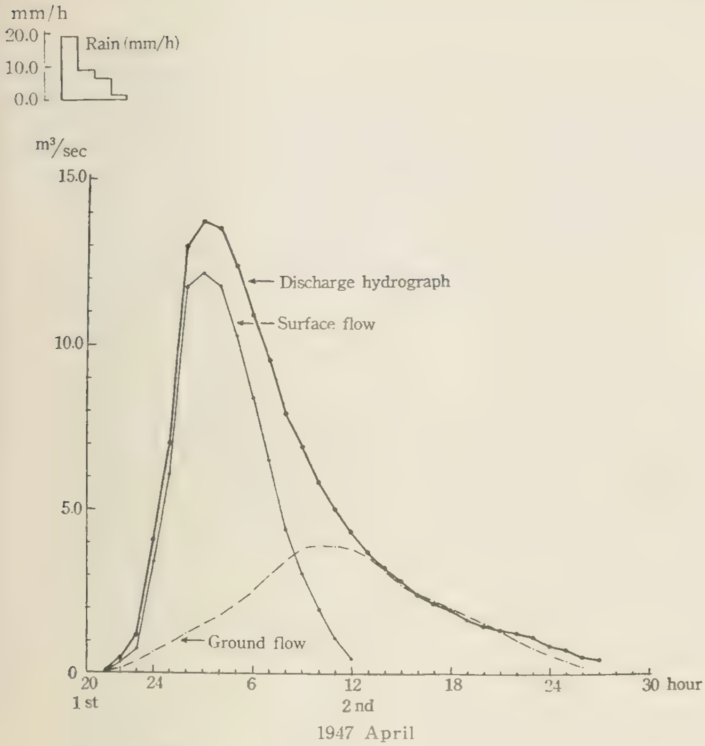


Fig. 6B Hydrograph for continuous rain

when the surface flow terminates. On the other hand, since the ground flow discharge becomes almost stationary after about 12 hours from the beginning of the discharge as is seen from Fig. 5, we can say that the ground water discharge increases in the manner of Fig. 5 for 12 hours from the beginning. Thus we can determine the ground water hydrograph both for one hour rainfall and for continuous rainfall, and consequently we can obtain the surface runoff hydrograph by subtracting the ground flow from the discharge hydrogra-

Table 3.

Date of rain	R (mm)	t_d (hour)	R/t_d (mm/h)	Q_r (mm)	Q_s (mm)	I (mm)	I/t_d (mm/h)
12, 13 Feb. 1944	53.3	14	3.8	2.7	9.3	44.0	3.14
5, 6 Mar. 1949	28.5	7	4.1	2.3	4.6	24.9	3.56
4, 5, 6 Apr. 1946	53.3	9	5.9	2.2	9.0	44.3	4.92
	60.5	16	3.8	11.0	22.5	38.0	2.38
30, Apr. 2, Mar. 1946	183.6	46	4.0	35.0	70.6	113.0	2.46
19, 20 Mar. 1949	127.6	18	7.1	36.1	38.9	88.7	4.93
27, 28 Mar. 1949	46.9	11	4.3	11.0	6.0	40.9	3.72
4, 5 June 1944	109.1	17	6.4	30.7	52.6	56.5	3.32
13, 14 June 1948	105.2	29	3.6	3.0	8.6	96.6	3.33
15, 16 June 1949	70.9	13	5.5	38.0	12.0	58.9	4.53
3, 4 July 1948	84.0	10	8.4	22.6	46.5	37.5	3.75
24, Dec. 1949	24.1	5	4.8	1.7	3.0	21.1	4.22

ph (i. e., observed hydrograph minus base flow hydrograph). An example for one hour heavy rain is shown in Fig. 6A and that for continuous rain is shown in Fig. 6B.

With respect to total rainfall **R** mm, duration of rainfall t_d hours, total ground water discharge Q_g mm, total surface runoff Q_s and total infitration **I** mm,several examples are given in Table 3. Also in Figs. 7–9 the relations between those quantities are shown.

According to the data on April 4–6, 1946, in Table 3, it is clear that the surface runoff due to the present rainfall is large and the infiltration (containing initial loss) is small in the case when it has rained just previously.

Further, we can conclude from Figs. 7–9 as follows: For 5–46 hours rain

1) The total infiltration and the total surface runoff increase as the total rainfall increases.
2) The total ground water discharge increases with the total rainfall until it reaches about 100 mm, and does not change so much in the rainfall of 100–180 mm.
3) In the case when the total rainfall is more than 100 mm, the discrepancy between the increasing of the infiltration and the constancy of the ground water discharge will be accounted for by deep seepage, etc.

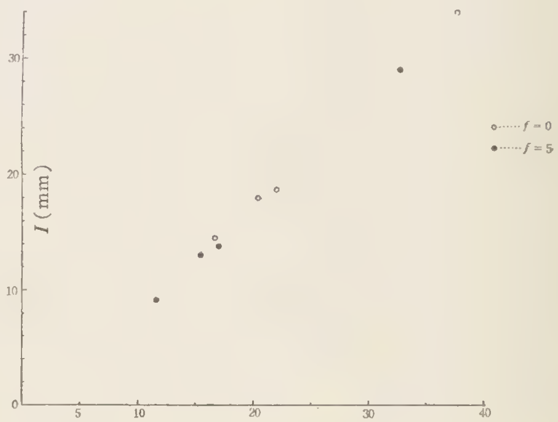


Fig. 7 **I**~(**R**–f) Diagram for one hour rain

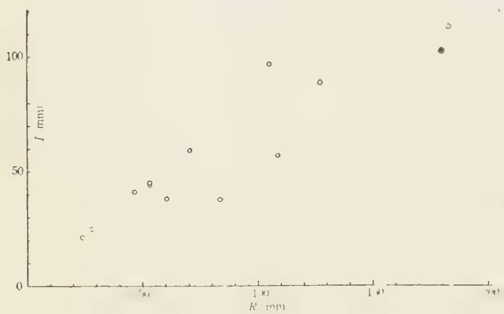


Fig. 8A **I**–**R** Diagram for continuous rain

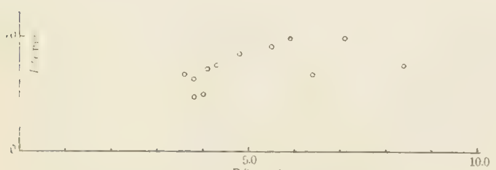


Fig 8B **I**/t_d–**R**/t_d Diagram for continuous rain

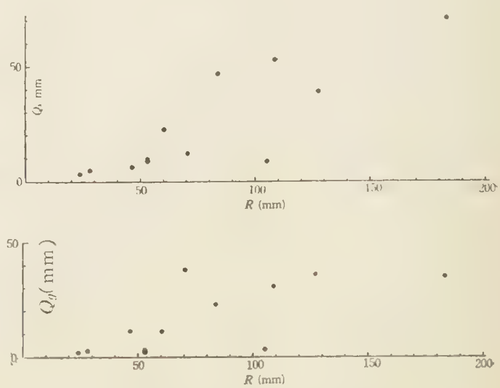


Fig. 9 Q_s–**R** Diagram
Q_g–**R** Diagram

4. Unit-hydrograph of Surface Flow

By the method explained in the last section (cf. Fig. 6A), we obtain surface flow hydrograph due to one hour rain by subtracting the ground water discharge from the discharge hydrograph. This hydrograph represents Q in (2). Dividing this hydrograph, namely the expression (2), by $[R-f-I]$, we will get the hydrograph

$$Q' = \int_0^t G\delta(\tau)d\tau - \int_1^t GI'd\tau, \qquad t > 1 \quad \cdots\cdots (4)$$

where $\qquad Q' = \frac{Q}{[R-I-f]}, \quad I' = \frac{I}{[R-f-I]}$

In order to obtain the unitgraph from (4), we must first know I and f, which can not be determined directly from the observed data. We can obtain (total infiltration $+f$) by subtracting the ground water discharge from the precipitation, and can estimate f from the observed data of weak rain which does not raise the stage at all. Therefore the question is how to determine the time distribution of the infiltration I. As is seen from the hydrograph of the ground flow in Fig. 5, the flow is stationary for some time after the lapse of 12 hours from the beginning of the runoff. During this period, the ground water discharge must balance with the supply of ground water. Therefore we can say that the infiltration ceases when the ground water discharge begins to decrease. This is just the time when the surface runoff terminates as is seen from Fig. 6A. This fact shows that the infiltration keeps up with the surface runoff. Fig. 6A shows that the duration of surface flow is about 15 hours and the peak of surface flow takes place after 5 hours from the beginning of the runoff. This holds for all observed one hour heavy rains. If we apply the relations above described, the time when the rain ceases in expression (2) is considered to correspond to the peak time of the present case. In other words, one hour precipitation correspo-nds to 5 hours runoff in hygrograph. After that time, only the surface detention contributes to the surface flow which keeps on for 10 hours. Thus surface detention is considered to keep on for two hours after the rain is over in (2). Therefore the infiltration which must keep up with the surface detention is considered to continue for 2 hours after the rain stops; that is, the total duration of the infiltration for one hour rain should be considered to be 3 hours in (2). The rate of infiltration while it is raining is expected to be larger than that after the rain is over, but it is difficult to decide this ratio *a priori*.

According to the experiment and observation so far made, the rate of infiltration decreases exponentially with time if the duration of rain is short, i. e.

$$I = I_0 e^{-\alpha t} \quad \cdots\cdots (5)$$

As the expression (5) contains two unknown quantities, I_0 and α, two conditions are necessary to determine them. One of them is given by the total infiltration S, which is obtained by integrating (5) from $t=0$ to $t=3$, namely;

$$S = \frac{I_0}{\alpha}(1 - e^{-3\alpha}) \quad \cdots\cdots (6)$$

As is seen in Figs. 4 and 6A, the ground flow is approximately stationary at the end stage of surface flow due to one hour rain. This fact tells us that the supply of ground water is equal to its discharge. Because of the shortness of the duration of rainfall, the supply of the ground water is considered to be equal to the infiltration at that stage, namely, at $t=3$. Taking one hour as unit of time, and representing the rate of infiltration at $t=3$ with I_3, I_3 should be equal to the ground water discharge during 5 hours. Thus we obtain the other condition, i. e.

$$I_3 = I_0 e^{-3\alpha} \qquad (7)$$

Using (6) and (7), we estimated I_0 and α for three examples and computed the ratio of hourly infiltration to total infiltration. The results are as given in Table 4.

Table 4.

Date of rain		S (mm)	I_3 (mm/h)	I_0 (mm/h)	α	I/S Time interval 0–1	1–2	2–3
5, Aug.	1948	13.05	1.0	12.85	0.85	0.62	0.27	0.11
23, Sep.	1948	12.77	1.0	11.36	0.81	0.61	0.27	0.12
26, Sep.	1948	8.55	0.6	8.16	0.87	0.63	0.26	0.11
					mean	0.62	0.27	0.11

In order to verify these results, we tried synthesis of the surface flow hydrographs by the unitgraph which is obtained under various assumptions for the ratio of hourly infiltration to total infiltration, and sought the values which give the best fit between the calculated hydrograph and observed one. The agreement between the best fit values and those given in Table 4 was satisfactory.

Next, the values of f should be different case by case, because it depends on the state of soil and vegital cover, especially on the previous rainfall. In our case, we put $f=6$ in the average, taking account of the examples of rainfall without any runoff.

Using the values above obtained, the hydrograph corresponding to (4) was calculated for three examples of one hour heavy rainfall (Aug. 5th, Sep. 23rd and 26th, 1948). The mean values of them are given in Table 5 and graphically represented in Fig. 10A.

Table 5.

Time (hour)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
(m^3/sec) $\times 10^{-2}$	3	17	62	145	210	167	123	91	67	49	32	18	12	5	4

On the other hand, using the time distribution of infiltration, we can rewrite (4) as follows,

$$Q_0 = \int_0^t G\delta(\tau)d\tau - \int_1^t GI'\delta(\tau-1)d\tau - \int_2^t GI''\delta(\tau-2)d\tau,$$

where

$$Q_0 = \frac{Q}{R-0.62S-6}, \quad I' = \frac{0.27S}{R-0.62S-6},$$

$$I'' = \frac{0.11S}{R-0.62S-6}$$

S: total infiltration, $f=6$.

Therefore, unitgraph is given by

$$\int_0^t G\delta(\tau)d\tau = Q_0 + \int_1^t GI'\delta(\tau-1)d\tau + \int_2^t GI''\delta(\tau-2)d\tau. \quad \cdots\cdots\cdots\cdots\cdots \quad (8)$$

Since G is an unknown function, we can not derive unitgraph directly from (8). But we can calculate it by a successive approximation, since the correction terms to Q_0 have the same form as the unitgraph. Putting

$$Q_1(t) = I'Q_0(t-1),$$

we get from (8)

$$\int_1^t GI'\delta(\tau-1)d\tau = Q_1 + \int_2^t GI'I'\delta(\tau-2)d\tau + \int_3^t GI''I'\delta(\tau-3)d\tau. \quad \cdots\cdots\cdots\cdots\cdots \quad (9)$$

Substituting (9) in (8), we get

$$\int_0^t G\delta(\tau)d\tau = Q_0 + Q_1 + \int_2^t G[I''+I'I']\delta(\tau-2)d\tau + \int_3^t GI''I'\delta(\tau-3)d\tau. \quad \cdots\cdots \quad (10)$$

Next, putting $Q_2(t) = (I''+I'I')Q_0(t-2)$, we get

$$\int_2^t G[I''+I'I']\delta(\tau-2)d\tau = Q_2 + \int_3^t G[I''+I'I']I'\delta(\tau-3)d\tau$$

$$+ \int_4^t G[I''+I'I']I''\delta(\tau-4)d\tau \quad \cdots\cdots\cdots\cdots\cdots \quad (11)$$

Substituting (11) in (10), we get

$$\int_0^t G\delta(\tau)d\tau = Q_0 + Q_1 + Q_2 + \int_3^t G[I''I' + (I''+I'I')I']\delta(\tau-3)d\tau$$

$$+ \int_4^t G[I''+I'I']I''\delta(\tau-4)d\tau.$$

In the same manner, we get finally

$$\int_0^t G\delta(\tau)d\tau = Q_0 + Q_1 + \cdots\cdots + Q_n + \int_{n+1}^t G\triangle I_{n+1}\delta(\tau-n-1)d\tau$$

$$+ \int_{n+2}^t G\triangle I_n I''\delta(\tau-n-2)d\tau, \quad \cdots\cdots\cdots\cdots\cdots \quad (12)$$

where

$$\triangle I_n = I''\triangle I_{n-2} + I'\triangle I_{n-1},$$

$$\cdots\cdots\cdots\cdots$$

$$\triangle I_3 = I''\triangle I_1 + I'\triangle I_2,$$

$$\triangle I_2 = I'' + I'\triangle I_1,$$

$$\triangle I_1 = I',$$

$$\triangle I_0 = 1$$

and $$Q_n(t) = \triangle I_n Q_0(t-n)$$

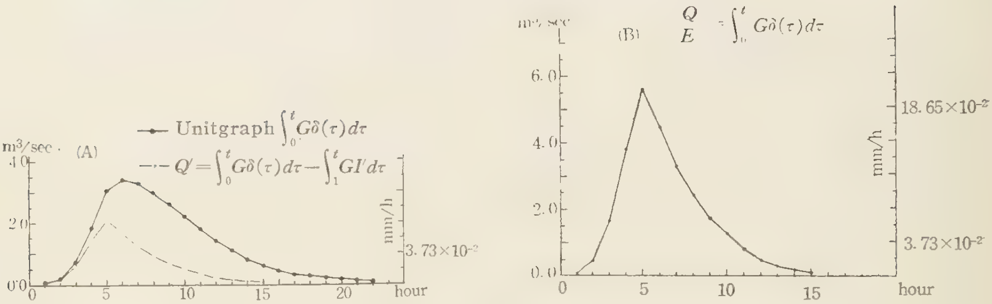


Fig 10 Unit hydrograph of surface flow

Using the values of Q_0 given in Table 5, we can easily obtain unitgraph $\int_0^t G\delta d\tau$. The result is shown in Fig. 10A and tabulated in Table 6.

For the sake of comparison, under the assumption that the infiltration ceases when rain stops we can derive surface flow hydrograph corresponding to (3′) from the same three examples as above used. Namely we divide the surface flow hydrograph for one hour rain by $[R-f-I]$ and take their average. The result is shown in Fig. 10B. In this case $(f+I)\equiv(f+S)$ is easily obtained. This procedure means that the actual hydrograph for surface flow is assumed to be represented by the expression (3′), that is

$$Q' = \int_0^t G\delta(\tau)d\tau, \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (13)$$

where

$$Q' = \frac{Q}{[R-f-I]}\,.$$

Table 6.
Unitgraph

Time (hour)	$(m^3/sec)\times10^{-2}$
1	3
2	18
3	70
4	179
5	301
6	333
7	326
8	297
9	260
10	219
11	177
12	137
13	105
14	77
15	58
16	39
17	28
18	21
19	14
20	10
21	8
22	4

5. Infiltration

In order to synthesize the surface flow hydrograph for continuous rainfall by unitgraph, we must know the value of I in (1). By the method described in section 3, we obtain the total amount of surface runoff from the observed runoff and the ground water discharge. The total infiltration containing f, i. e. $(S+f)$, is obtained

by subtracting the total surface runoff from the total rainfall. Since f depends chiefly on the previous rainfall, we assume f as follows.

$f=0$ in case when it rains fairly on the previous day,

$f=5-10$ in case when it rains in the preceeding 7 days (taking the amount of rain and the season into consideration), and

$f>10$ in case when it does not rain at all in the preceeding seven days.

For one hour heavy rain, the relation between $R-f$ and I is shown in Fig. 7. For 5–46 hours rain, the relation between the total rainfall and the total infiltration including f is shown in Fig. 8. Judging from these figures, it seems that the relation between the total rainfall and the total infiltration is approximately linear.

Therefore we shall assume that the infiltration per hour is proportional to the rainfall per hour for the continuous rain. Then we get the following relation, in regard to the residual rainfall ($\mathbf{R}-f$):

$$I_i=\beta R_i, \quad \Sigma I_i=\beta\Sigma R_i \qquad (14)$$

where R_i: Amount of rain during one hour $(i-1)^h-i^h$ from the beginning of the residual rainfall, ($i=1,2,3,\cdots\cdots\cdots$)

I_i: Amount of infiltration during the same period,

β: Constant.

6. Synthesis of surface runoff hydrograph by the unitgraph

Now, we shall synthesize the surface runoff hydrograph, using (12) and (13). From (14), we get

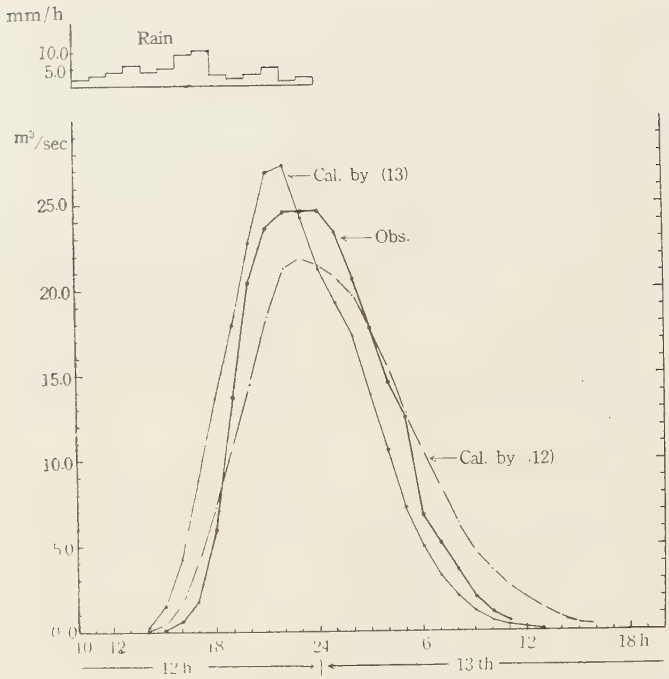
$$R(t)-I(t)=(1-\beta)R_i \quad (i-1<t\leq i) \qquad (15)$$

Substituting (15) in (1), we obtain the surface runoff Q, i. e.

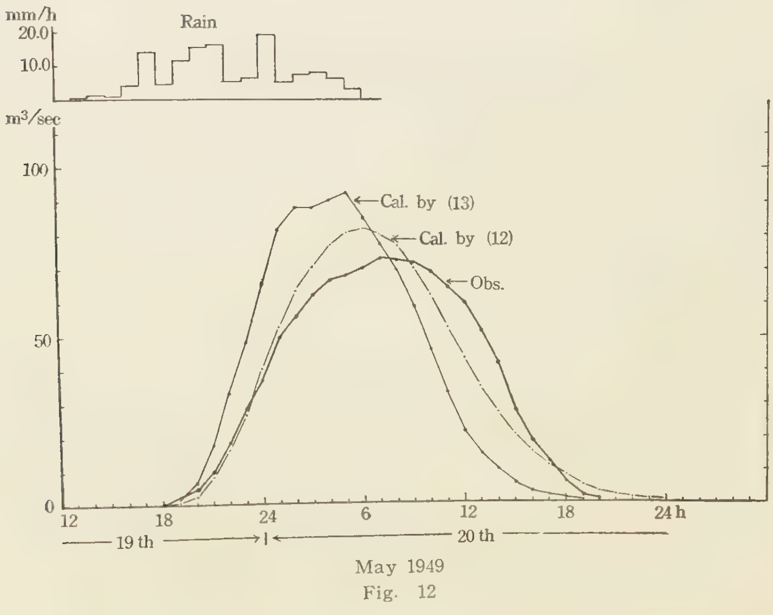
$$Q=\sum_i(1-\beta)R_i\int_{i-1}^{t}G\delta(\tau-i+1)d\tau \qquad (16)$$

Thus, by use of the values of $\int^t G\delta d\tau$ as given in Table 6, we can easily compute Q by the usual unitgraph method. In the case when one uses the expression (13) as the unitgraph, the procedure of calculation is quite similar.

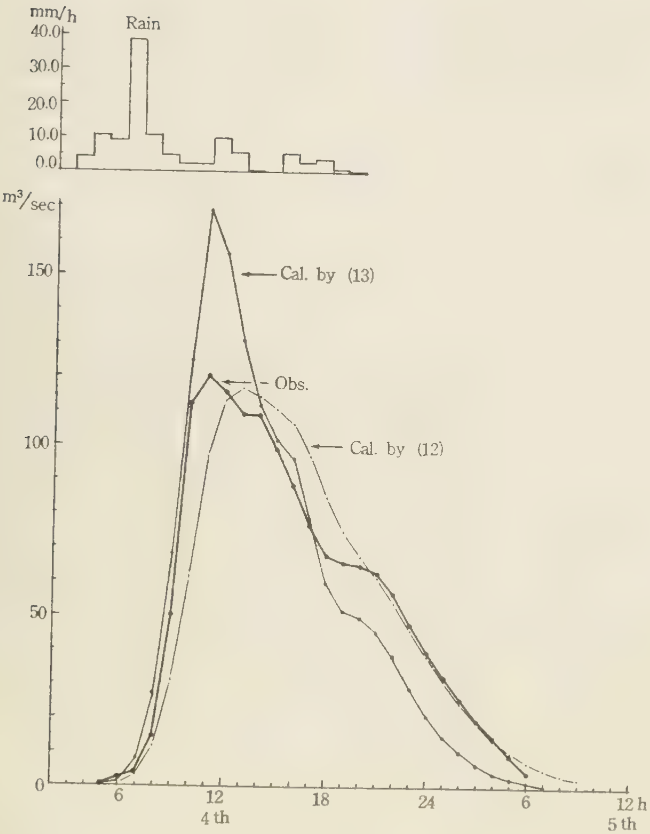
The results calculated for four examples are shown in Figs. 11–14. No complete agreement is found between the calculated values and the observed ones. But the calculated values obtained by using (12), namely, under the assumptions that the infiltration keeps up after the rainfall is over, are in better agreement with the observed ones than the values calculated by using (13), namely, under the assumption that the infiltration stops when rain stops. The standard deviations for both cases are given in Table 7 which show the difference of fitness of these two methods more clearly. In the latter case, there are tendencies that the peak time comes earlier and the runoff at the peak becomes too much, while in the former case, the peak takes



February 1944
Fig. 11



May 1949
Fig. 12



June 1944
Fig. 13

place almost at the same time as the observed one and the duration of the runoff agrees quite well with the observed one. The discrepancy of runoff may be caused by the non-linear relation. But, if we take the imperfection of the data of precipitation and $Q-H$ curve into consideration, we can say that the accordance between the observed hydrograph and calculated one by (12) is satisfactory.

Table 7

Example	$\sqrt{\frac{(\text{obs.}-\text{cal.})^2}{n}}$ for total runoff		$\sqrt{\frac{(\text{obs.}-\text{cal.})^2}{5}}$ for 5 hours near peak	
	used expression		used expression	
	(12)	(13)	(12)	(13)
Fig. 11	275	343	360	303
Fig. 12	802	2112	744	1397
Fig. 13	1347	1745	2167	3066
Fig. 14	1748	2889	2056	5056

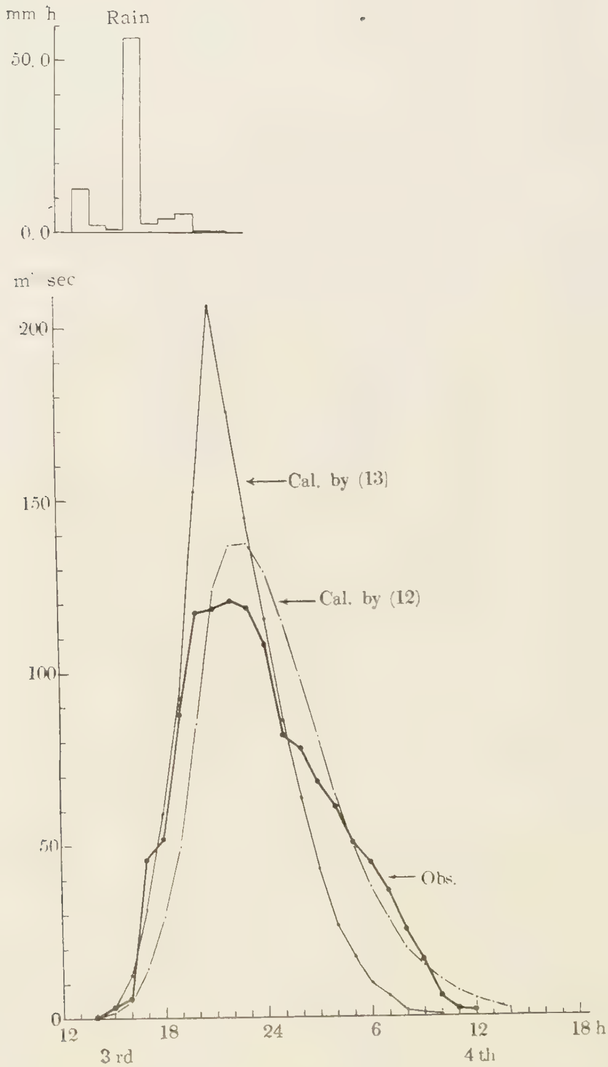


Fig. 14

Acknowledgment

The writer wishes to express his sincere thanks to Prof. S. Hayami, Kyoto University, for his kind guidance and encouragement, and to Prof. S. Tomotika for his encouragement during the course of this research and for his inspection of the

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Precipitation and water-level at Uchinomaki

	Mar. 1946			June 1947			
Day	13		14	1		2	3
Time	*R*	*H*	*H*	*R*	*H*	*H*	*H*
1			21.0			55.7	51.5
2			20.8			56.0	51.4
3			20.8			57.0	51.0
4			20.4			57.6	51.0
5			20.2			57.8	
6			19.8			58.3	
7			19.4			58.6	
8		21.0	18.2			58.6	
9	1.5	21.0	18.0			58.6	
10	9.0	21.0	17.6			58.6	
11		21.6	17.6			58.0	
12		24.5	17.6			57.5	
13		28.0	17.6			57.2	
14		32.2	17.4			56.2	
15		35.4	16.8			56.0	
16		35.3	16.8			55.8	
17		33.6				55.4	
18		31.4			53.0	54.6	
19		29.2		4.6	53.3	54.4	
20		27.0			54.0	53.5	
21		25.0			54.6	53.0	
22		24.0			55.2	52.6	
23		22.8			55.5	52.0	
24		21.7			55.6	51.7	

	Sep. 1948			Oct. 1948			
Day	9		10	20		21	
Time	*R*	*H*	*H*	*R*	*H*	*R*	*H*
1			61.2			1.3	51.0
2			60.6				51.8
3			60.6				51.0
4		61.8	59.7				50.7
5	1.9	61.9	59.1				50.6
6	0.9	60.8	58.8				49.9
7	3.7	61.7	58.8				50.0
8		61.0	59.3				50.2
9		61.2	58.4				50.3
10		61.4	57.8				50.3
11		62.0	57.4				50.2
12		63.0	57.3				51.0
13		62.5					51.2
14		62.2					50.6
15		62.0					51.1
16		62.8					50.5
17		63.0					50.4
18		63.1					50.4
19		63.0					50.0
20		62.7					49.8
21		63.0			49.6		49.7
22		62.5		2.6	50.3		49.4
23		62.5			50.4		49.6
24		62.1			51.1		49.5

ON ANNUAL AND SECULAR VARIATIONS OF THE HYDROSTATIC HEAD OF HOT SPRING (PART II)

Tosisato MUROTA

(Received November 1, 1958)

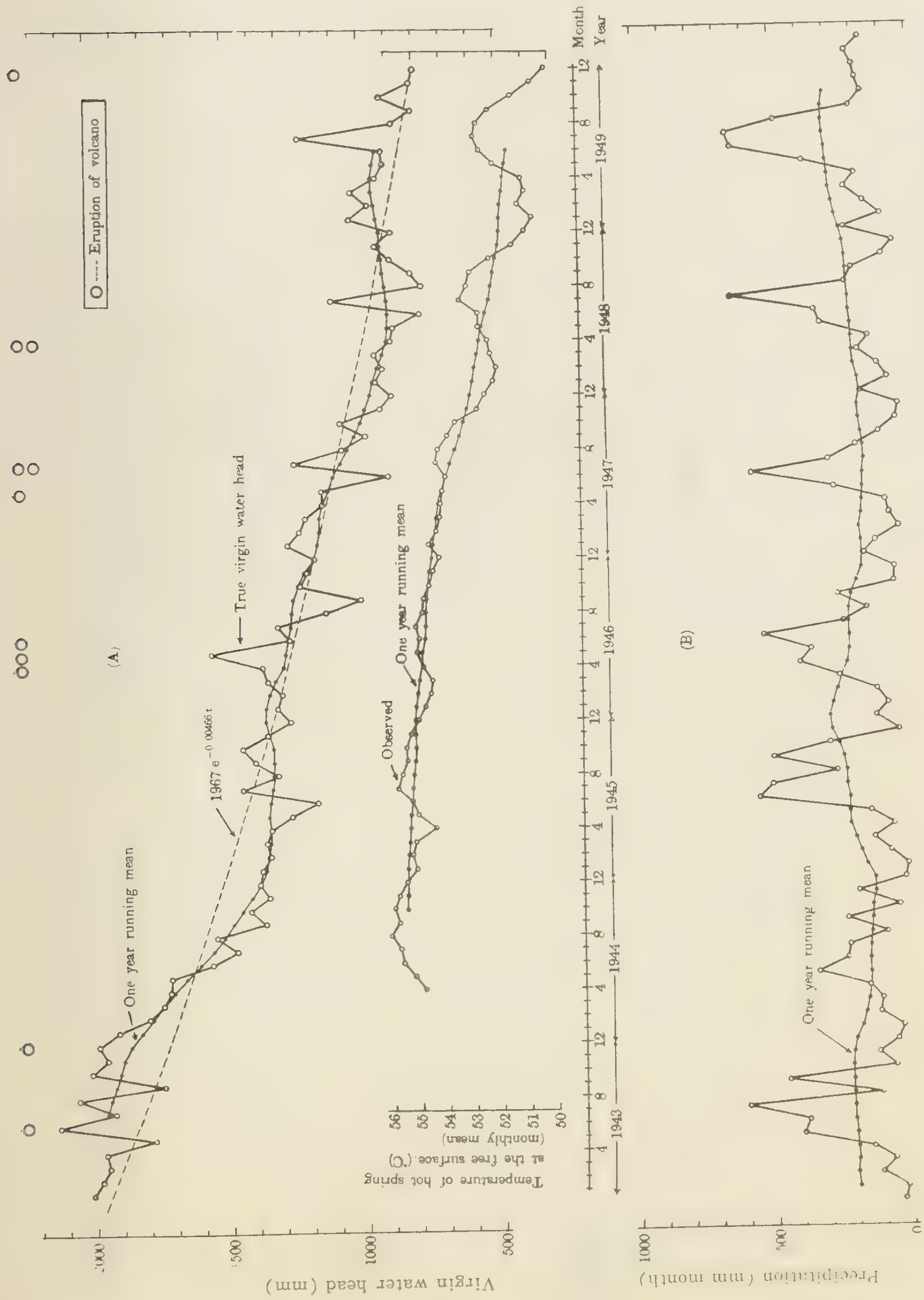
1. In a previous paper[1] with the same title, the writer studied the variations of the hydrostatic head of hot spring of Uchinomaki Observatory which belongs to the Volcano and Hot-Spring Research Institute of Kyoto University. Examining the effect of precipitation and atmospheric pressure on hydrostatic head of hot spring, he named the values that are considered to be affected no more by the precipitation and atmospheric pressure "True virgin water head". Their values are given in Table 10 of the previous paper. He also suggested that a part of the variations of the true virgin head may be related to the activity of volcano Aso.

In this paper the writer wishes to treat the variations in true virgin water head. The "True virgin water head" given in Table 10 is shown graphically in Fig. 1A. Assuming that the secular trend can be expressed by an exponential function, we find by the method of least squares, its form that is expressed by

$$1967\ e^{-0.00466t}\ \text{(in month, mm)}$$

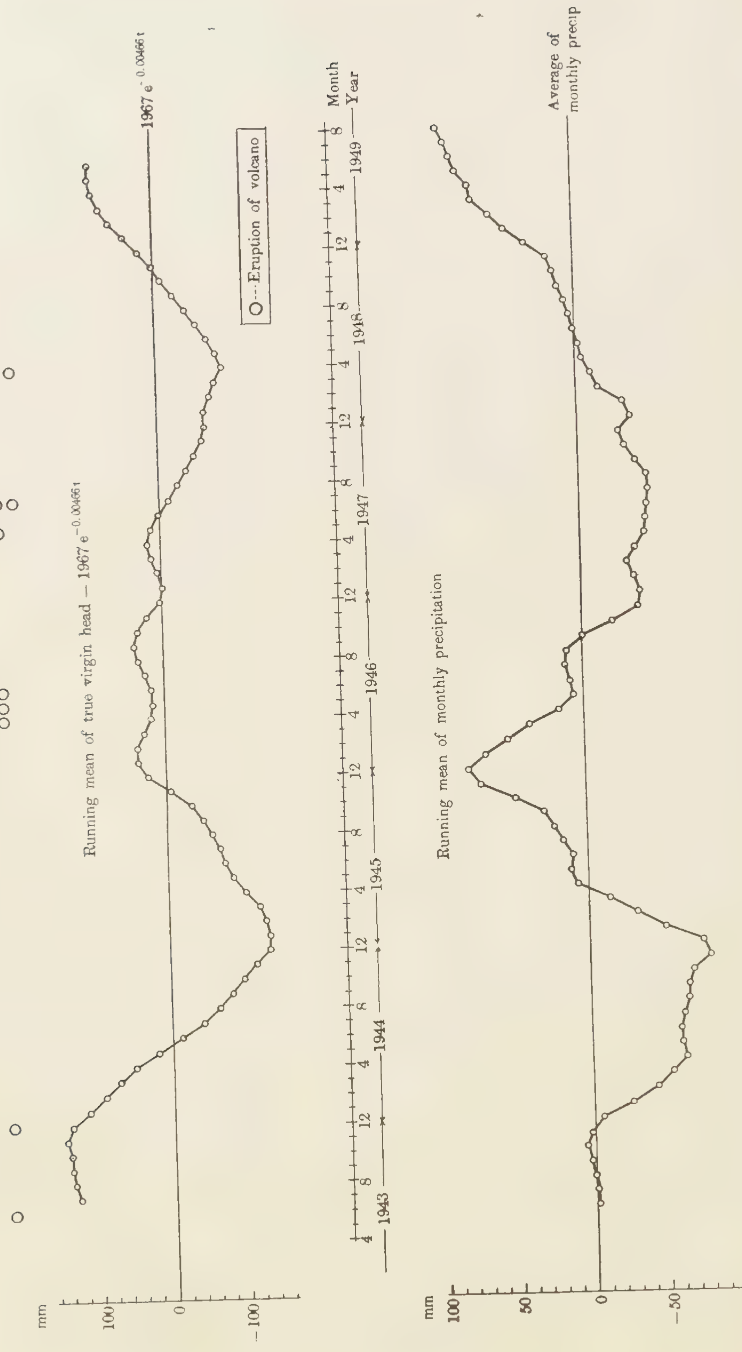
where t is time measured in unit of month and 1967 is the value in December 1942. Since original values fluctuate about this curve, the one-year running means of true virgin head were calculated. Both the exponential curve and the running means are shown in Fig. 1A, from which it will clearly be seen that the one-year running means show a regular deviation from the exponential curve.

For the sake of comparison, the one-year running means of the monthly precipitation at Uchinomaki in the same period, 1943–1949, were also calculated. The data of the precipitation are also given in Table 2 of the previous paper. The monthly precipitation and their one-year running means are shown in Fig. 1B. Comparing Fig. 1A with Fig. 1B, we see that the two curves of the running means have similar forms and show a variation of long period respectively. These variations of long period of the two curves are extracted and shown in Fig. 2. It is seen that the forms of these two curves are very similar to each other. It is worth noticing that both curves show a change with period of 3–3.5 years and especially that the phase of the variation of head curve lags by about 0.5 year behind that of precipitation. As treated in the previous paper, the effect of the monthly precipitation on the head does not show any phase lag. Thus, we may infer that the mechanism of the effect of precipitation seen in Fig. 2 that gives about 0.5 year phase lag must differ from that of the effect of precipitation stated in the previous paper which does not give phase lag. Since the source of hot spring is artesian ground water, the variations of head should propagate rapidly all over the artesian water zone. On the other hand,



O ---- Eruption of volcano
(A)
True virgin water head
One year running mean
$1967\,e^{-0.00466\,t}$
Observed
One year running mean
Virgin water head (mm)
2000
1500
1000
500
Temperature of hot spring at the free surface (°C) (monthly mean)
56
55
54
53
52
51
50
Month
Year
1943
1944
1945
1946
1947
1948
1949
(B)
One year running mean
Precipitation (mm month)
1000
500
0

Fig. 1A and 1B



Running mean of true virgin head — 1967 e^−0.00466t
1967 e^−0.00466t
○ ··· Eruption of volcano
Month
Year
1943
1944
1945
1946
1947
1948
1949
Running mean of monthly precipitation
Average of monthly precip
mm
100
0
−100
mm
100
50
0
−50
−100

Fig. 2

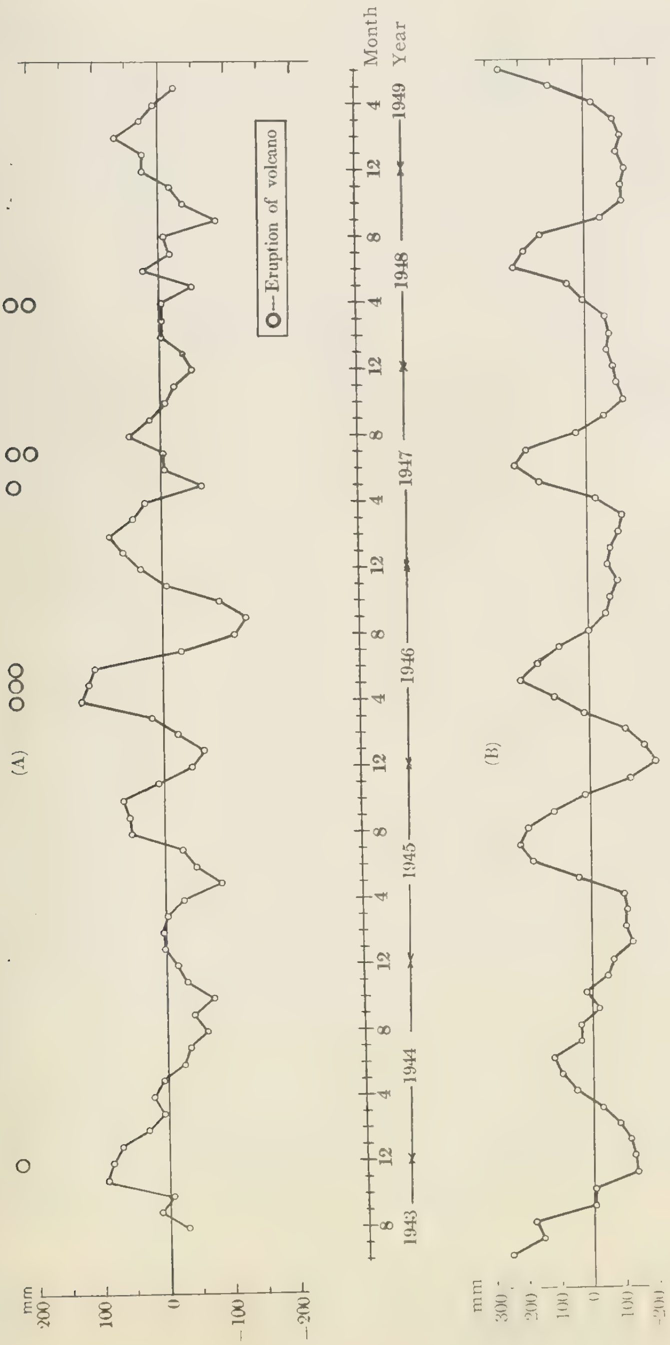

(A)
O— Eruption of volcano
mm
200
100
0
−100
−200
Month
Year
1943
1944
1945
1946
1947
1948
1949
(B)
mm
300
200
100
0
100
200

Fig. 3A and 3B

the effect of precipitation on the head is caused by the increase of water-pressure due to the infiltration of precipitation. Therefore it is concluded that the effect of precipitation that gives no phase lag has its cause in a shallow zone, while the effect that gives a phase lag has its cause in a deep zone.

Since the interval of the observation is rather short and the representation of volcanic activity is difficult, no definite conclusion can be drawn yet, but it is of interest to note that, though it may be an accident, the eruption of volcano Aso mostly occured as shown in Figs. 1 and 2, in the period when the head is high in the variation of long period.

2. Even though the above-mentioned variation of long period is removed, there remain peculiar variations in virgin head. The residual of the true virgin head subtracted by the one-year running mean shows irregular variation of short period. The three-months running means of the residual were calculated and are represented graphically in Fig. 3A. For the sake of comparison, the three-months running means of the similar residuals of the monthly precipitation from the one-year running mean are shown in Fig. 3B.

As seen from Figs. 3A and 3B, the curve for the virgin head shows a variation of about 7.5 months period in the mean, and, on the other hand, the curve for the monthly precipitation shows distinctly the yearly variation but no variation of the same period as the head.* The fact that the curve for the virgin head shows no yearly variation means that the effect of the monthly precipitation is removed perfectly. From this point of view the variation of about 7.5 months period appeared in the head cannot be attributed to the precipitation. The head of hot spring is affected also by atmospheric pressure. However, it must be said that the effect of atmospheric pressure as well as precipitation is removed enough since the atmospheric pressure has distinctly the yearly variation as stated in the previous paper.

We can not think of the other exogenous cause of the variation of the head of hot spring except atmospheric pressure and precipitation, therefore we have no way of thinking but to consider that it is due to endogenous cause. However, we do not know at all what is the nature of this endogenous cause. But in connection with this point it is an interesting fact that the period of time, when this peculiar variation of 7.5 months period appears remarkably, is in accordance with the period of time, when the head is high in the variation of long period stated in the last section, and in the peak time of the variation of 7.5 months period the eruptions of volcano Aso occured oftener than the other time. In the period of time when the head is low in the variation of long period, the variation of 7.5 months period does not remarkably appear or at all.

* The variation of about 7.5 months period is also clearly seen with opposite phase in the one-year running means. It is caused by taking one-year running means to the variation of 7.5 months period. In general, the reducing factor of a 7.5 months variation in the one year running means is negative and is given by about −0.073.

3. As already noticed, the general trend of the virgin water head decreases exponentially. Since the hot spring at Uchinomaki Observatory where the head variation was observed is 153 m in depth and was the only one hot spring that bored till this depth. So this general trend can not be attributed to the scoop by other hot springs from the same water vein. In Uchinomaki area there are several other water veins from which hot springs were supplied at that time. Because of the use for a long time, the amount of gushing is drying up and lately the same water vein as Uchinomaki Laboratory's spring is gradually developed as the source of hot springs. A water vein is never isolated from the adjacent water veins and it is usual that they are connected to each other. Therefore if the water pressure of the upper water vein decreases, the ground water of the lower water vein will infiltrate into the upper vein and the pressure of the lower vein will decrease. The general trend of the virgin head is considered to be due to such a decrease of the water pressure of the upper veins. If this inference is true, the water pressure of the lower vein will increase as soon as that of the upper vein increases by some cause. The cause of the precipitation effect which does not show any phase lag has been inferred to lie in the shallow region. The fact that it immediately gives rise to increase of the virgin head seems to support the above inference. As stated in the previous paper, the damping factor of the effect of every monthly precipitation on the head is expressed by the form,

$$e^{-0.34t},\ t \text{ in months.}$$

This fact is also understood by considering that the increase of water pressure occurs in the shallow layer in which the ground water easily moves. The damping factor of the base flow of the Kurokawa is given by

$$e^{-0.55t},\ t \text{ in months.}$$

4. The variation of the surface temperature of the hot spring at Uchinomaki Observatory is interesting in connection with the variation of the head. It is shown in Fig. 1A. Temperature varies in accordance with the observed head of the hot spring which is not corrected for the effects of both precipitation and atmospheric pressure, as is usually observed in other hot springs(2). The rate of cooling along a hot spring pipe is controlled by the rate of its discharge and the latter in turn is determined by the head of hot spring, hence the surface temperature of a hot spring is controlled by its head, and temperature rises when head increases. In connection with this relation between temperature and head, it will be worth to mention that during the period when the variation of 7.5 months period is conspicuous, the annual variation of temperature does not develop so well, in spite of large annual variation of precipitation. It seems that the normal relation between temperature and head is reversed with respect to the variation of 7.5 months period in the head.

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Temperature of Hot Spring at the free Surface (monthly mean in 10^{-2} °C)

1944

Day \ Month	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
1		5525	5595	5575	5615	5611	5570	5595	5540
2				5615	5535	5630	5576	5570	5545
3		5480		5575	5595	5615	5575	5570	5570
4		5495	5545	5582	5555	5590	5610	5580	5560
5		5523	5555	5620	5600	5570	5588	5675	5540
6		5495	5575	5550		5565	5588	5515	5565
7		5528	5530	—		5575	5610	5615	5570
8		5517	5575	5555	5610	5580	5620	5610	5560
9		5511	5595	5535	5650	5570	5620	5585	5555
10		5438	5615	—	5570	5531	5598	5590	5570
11		5480	5575	—	5607	5530	5585	5570	5530
12		5520	5575	—	5585	5550	5600	5560	5550
13		5525	5540	5620	5620	5570	5565	5595	5555
14	5435	5525	5590	5630	5580	5580	5585	5510	5540
15	5475	5525	5620	5600	5590	—	5570	5520	5565
16	5445	5515	—	5590	5655	5590	5575	5575	5555
17	5470	5525	5450	—	5575	5585	—	5575	5545
18	5430	5525		5630	5611	5568	5610	5585	5540
19	5482	5540	5528	5575	5621	5576	5630	5590	5540
20	5473	5530	5570	5555	5630	5582	5620	5590	5540
21	5478	5545	5581	5525	5565	5593	5550	5570	5540
22	5490	5535	5570	5530	5675	5580	5600	5550	5555
23	5500	5535	5551	5540	5560	5608	5580	5570	5540
24	5506	5545	5600	5570	5645	5569	5600	5560	5555
25	5495	5550	5590	5520	5610	5595	5570	5575	5545
26	5495	5525	5569	5590	5651	5570	5580	5569	5535
27	5493	5520	5575	5545	5620	5550	5590	5572	5550
28	5540	5560	5609	5620	5630	5568	5610	5585	5555
29	5512	5550	5542	5595	5600	5600	5572	5580	5550
30	5520	5555	—	5555	5595	5583	5610	5590	5550
31		5575		5591	5621		5595		5560
Mean	5485	5524	5569	5576	5606	5579	5592	5577	5551

1945

Day \ Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct	Nov.	Dec.
1	5550	5530	5505	5510	5491	5510	5560	5590	5540	5560	5550	5500
2	5550	5545	5520	5450	5530	5537	5555	5595	5530	5530	5545	5515
3	5560	5535	5521	5400	5467	5500	5565	5600	5550	5550	5540	5505
4	5562	5475	5540	5420	5480	5496	5560	5580	5510	5510	5550	5510
5	5530	5555	5515	5510	5485	5430	5590	5590	5520	5550	5560	5510
6	5525	5530	5500	5510	5475	5498	5570	5545	5560	5555	5540	5510
7	5550	5545	5490	5520	5480	5410	5580	5565	5550	5580	5545	5510
8	5540	5530	5490	5380	5520	5510	5560	5570	5550	5560	5525	5525
9	5530	5515	5505	5420	5520	5500	5565	5585	5515	5530	5525	5500
10	5555	5510	5500	5520	5500	5541	5550	5560	5540	5550	5520	5520
11	5520	5510	5525	5510	5475	5525	5555	5580	5560	5520	5545	5510
12	5570	5510	5500	5420	5478	5519	5550	5550	5560	5560	5540	5510
13	5530	5520	5525	5540	5520	5500	5530	5550	5540	5555	5525	5505
14	5515	5512	5535	5105	5480	5470	5560	5500	5560	5560	5540	5510
15	5550	5550	5535	5200	5519	5520	5560	5518	5550	5530	5530	5500
16	5550	5530	5528	5200	5515	5505	5580	5550	5505	5560	5540	5500
17	5520	5515	5510	5525	5526	5560	5575	5570	5520	5560	5520	5500
18	5530	5520	5530	5200	5509	5525	5595	5570	5540	5570	5545	5500
19	5500	5520	5520	5510	5535	5580	5595	5560	5550	5560	5520	5490
20	5530	5500	5520	5480	5525	5560	5575	5550	5550	5560	5520	5485
21	5510	5500	5510	5490	5525	5560	5600	5565	5550	5560	5515	5500
22	5545	5545	5530	5500	5520	5570	5580	5560	5550	5560	5530	5500
23	5535	5513	5510	5480	5510	5565	5608	5545	5565	5560	5510	5480
24	5100	5513	5520	5470	5510	5545	5590	5550	5540	5535	5500	5490
25	5500	5510	5520	5515	5500	5560	5595	5550	5540	5550	5545	5500
26	5450	5550	5550	5460	5509	5575	5595	5550	5540	5540	5520	5490
27	5445	5520	5500	5483	5510	5565	5570	5550	5550	5525	5510	5495
28	5445	5585	5485	5470	5530	5570	5580	5550	5560	5500	5520	5480
29	5530		5515	5495	5510	5565	5582	5530	5540	5550	5520	5480
30	5525		5515	5512	5519	5565	5600	5540	5560	5510	5520	5490
31	5535		5520		5515		5590	5550		5545		5490
Mean	5512	5525	5516	5441	5506	5528	5575	5560	5543	5547	5531	5500

1 9 4 6

Month / Day	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
1	5480	5470	5450	5480	5410	5510	5520	5480	5485	5470	·	5435
2	5480	5460	5460	5480	5440	5510	5520	5490	5480	5480		5435
3	5483	5460	5450	5500	5500	5500	5510	5500	5480	5475	5462	5420
4	5485	5500	5460	5490	5510	5500	5500	5490		5460	5475	5430
5	5480	5460	5420	5500	5515	5510	5470	5490	5475	5440	5465	5445
6	5480	5460	5460	5510	5510	5500	5510	5485	5475	5455	5450	5425
7	5475	5465	5450	5470	5515	5510	5510	5490	5495	5465		5420
8	5485	5460	5450	5470	5510	5490	5530	5480	5482	5475	5450	5410
9	5489	5460	5420	5480	5525	5500	5500	5475	5480	5465	5455	5380
10	5472	5450	5460	5500	5515	5470	5485	5475	5475	5475	5470	5398
11	5500	5430	5420	5490	5520	5480	5510	5477	5470	5470	5455	5398
12	5520	5455	5430	5490	5510	5490	5500	5480	5485	5470	5450	5385
13	5470	5450	5465	5490	5415	5500	5520	5490	5470	5465	5455	5390
14	5465	5440	5430	5480	5480	5490	5525	5480	5470	5465	5420	5395
15	5460	5450	5450	5485	5510	5490	5500	5490	5485	5470	5403	5405
16	5470	5450	5450	5510	5500	5430	5510	5490	5470	5470	5432	5380
17	5460	5510	5450	5490	5520	5470	5510	5490	5465	5460	5420	5380
18	5430	5440	5430	5480	5515	5490	5510	5490	5470	5445	5440	5395
19	5475	5420	5450	5480	5520	5510	5520	5470	5460	5455		5400
20	5470	5430	5440	5490	5500	5470	5500	5500	5480	5460		5390
21	5465	5450	5460	5500	5520	5500	5510	5450	5460	5455		5425
22	5450	5440	5395	5480	5520	5480	5530	5470	5465	5460		5445
23	5460	5450	5450	5490	5505	5510	5500	5500	5450	5440	5445	5450
24	5470	5455	5460	5490	5510	5510	5510	5490	5565	5460	5440	5440
25	5460	5465	5460	5425	5510	5510	- -	5495	5570	5450	5430	5450
26	5485	5460	5470	5470	5510	5520	5510	5495	5465	5450	5440	5455
27	5475	5460	5465	5425	5510	5520	5515	5490	5475	5455	5450	5475
28	5460	5460	5470	5480	5510	5510	5590	5485	5470	5440	-	5465
29	5510		5480	5490	5500	5490	5495	5480	5460	5430	-	5450
30	5465		5475	5460	5500	5550	5480	5460	5462	-	5435	5455
31	5450		5460		5520		5490	5490				5455
Mean	5474	5456	5448	5483	5501	5497	5510	5484	5479	5460	5445	5421

1 9 4 7

Month / Day	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
1	5460	5445	5420	5430	5380	5410	5420	5430	5380	5360	5320	5260
2	5450	5439	5440	5425	5380	5415	5420	5425	5410	5360	5320	5260
3	5460	5435	5430	5425	5390	5410	5410	5430	5415	5365	5310	5260
4	5460	5435	5405	5420	5401	5410	5415	5420	5400	5370	5310	5250
5	5475	5460	5405	5425	5401	5405	5420	5420	5400	5375	5290	5265
6	5460	5440	5435	5425	5399	5410	5422	5425	5405	5370	5310	5265
7	5465	5435	5415	5420	5400	5410	5445	5420	5425	5360	5280	5240
8	5470	5435	5410	5420	5405	5405	5410	5415	5400	5365	5270	5240
9	5460	5425	5420	5425	5410	5395	5410	5415	5410	5365	5270	5270
10	5465	5435	5415	5420	5410	5390	5380	5415	5410	5365	5260	5270
11	5485	5435	5410	5420	5415	5415	5400	5415	5390	5350	5270	5265
12	5465	5435	5415	5420	5415	5415	5425	5395	5385	5355	5275	5255
13	5450	5430	5420	5420	5415	5410	5425	5420	5375	5355	5270	5220
14	5460	5435	5420	5420	5225	5385	5425	5395	5370	5360	5275	5225
15	5460	5435	5410	5420	5420	5385	5420	5415	5390	5365	5270	5225
16	5465	5435	5420	5420	5425	5380	5430	5390	5380	5360	5270	5255
17	5460	5435	5420	5420	5430	5380	5415	5390	5380	5360	5260	5230
18	5465	5425	5420	5418	5425	5380	5410	5390	5375	5370	5260	5220
19	5460	5420	5420	5415	5415	5380	5420	5420	5375	5365	5255	5245
20	5460	5420	5430	5420	5420	5395	5435	5440	5390	5365	5250	5245
21	5460	5415	5415	5415	5425	5380	5435	5445	5385	5360	5245	5210
22	5460	5420	5400	5415	5420	5380	5415	5465	5380	5360	5270	5250
23	5455	5415	5410	5399	5425	5380	5445	5445	5385	5360	5250	5250
24	5460	5420	5400	5395	5420	5390	5475	5440	5385	5360	5265	5250
25	5445	5420	5420	5398	5410	5390	5460	5430	5380	5365	5265	5230
26	5455	5425	5420	5398	5415	5390	5440	5425	5370	5360	5260	5230
27	5440	5420	5420	5400	5425	5410	5460	5432	5370	5325	5265	5225
28	5445	5420	5420	5380	5415	5405	5460	5430	5375	5325	5265	5225
29	5445		5420	5380	5410	5400	5440	5430	5390	5315	5260	5260
30	5445		5430	5375	5410	5425	5425	5390	5365	5315	5260	5260
31	5443		5430		5439		5425	5425		5310		5200
Mean	5458	5430	5418	5413	5406	5398	5427	5421	5388	5355	5273	5244

1948

Month / Day	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
1	5250	5200	5180	5220	5255	5265	5310	5280	5305	5260	5230	5110
2	5220	5230	5180	5255	5265	5265	5305	5275	5300	5255	5200	5120
3	5250	5220	5200	5245	5255	5265	5335	5320	5295	5260	5200	5115
4	5235	5190	5216	5240	5260	5260	5335	5315	5305	5260	5250	5110
5	5235	5190	5210	5230	5260	5260	5300	5330	5310	5255	5220	5080
6	5225	5190	5200	5225	5290	5260	5355	5330	5295	5250	5200	5130
7	5225	5190	5200	5220	5250	5260	5355	5335	5295	5240	5180	5120
8	5220	5185	5215	5225	5275	5265	5360	5330	5285	5230	5190	5060
9	5220	5180	5215	5220	5275	5265	5360	5320	5305	5230	5180	4980
10	5230	5190	5180	5255	5275	5260	5340	5320	5305	5200	5180	5170
11	5245	5170	5200	5245	5250	5260	5280	5310	5320	5200	5130	5150
12	5220	5160	5210	5230	5255	5265	5350	5320	5345	5210	5090	5080
13	5220	5200	5260	5240	5265	5265	5340	5310	5325	5210	5090	5120
14	5225	5220	5220	5230	5255	5230	5360	5300	5315	5230	5070	5090
15	5205	5220	5200	5235	5255	5260	5360	5290	5320	5220	5080	5070
16	5205	5220	5225	5240	5265	5290	5350	5310	5300	5300	5080	5050
17	5210	5200	5265	5230	5265	5300	5350	5310	5320	5240	5100	4920
18	5210	5200	5260	5230	5280	5230	5350	5315	5320	5220	5090	5070
19	5160	5210	5260	5230	5270	5290	5360	5325	5300	5220	5140	5090
20	5210	5195	5235	5225	5250	5285	5300	5325	5300	5210	5140	5080
21	5200	5230	5250	5225	5270	5285	5310	5310	5290	5210	5080	5100
22	5225	5240	5250	5230	5280	5250	5320	5300	5290	5200	5100	5130
23	5220	5225	5250	5220	5275	5285	5345	5300	5280	5200	5110	5135
24	5205	5175	5245	5220	5285	5290	5330	5300	5270	5180	5110	5130
25	5225	5200	5245	5230	5270	5280	5340	5290	5260	5200	5170	5200
26	5220	5210	5180	5230	5280	5270	5340	5295	5270	5200	5150	5130
27	5200	5230	5240	5245	5270	5295	5295	5300	5270	5180	5110	5110
28	5210	5200	5235	5235	5285	5300	5290	5300	5260	5180	5130	5100
29	5170	5200	5200	5230	5270	5290	5300	5305	5250	5170	5080	5110
30	5185		5240	5255	5265	5230	5290	5300	5270	5240	5110	5080
31	5180		5245		5265		5395	5295		5240		5090
Mean	5215	5202	5223	5233	5267	5269	5333	5309	5296	5223	5140	5098

1949

Month / Day	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
1	5110	5130	5110	5120	5140	5280	5260	5280	5263	4845	5061	5048
2	5140	5130	5100	5110	5130	5280	5260	5280	5275	5170	5070	4995
3	5100	5140	5100	5110	5100	5280	5260	5280	5261	5150	5085	5075
4	5120	5135	5020	5110	5120	5260	5245	5270	5250	5160	5115	5075
5	5120	5120	5070	5135	5140	5265	5245	5280	5265	5168	5105	5001
6	5070	5120	5090	5110	5170	5260	5260	5270	5260	5170	5065	5001
7	5060	5125	5060	5120	5160	5270	5255	5260	5250	5165	5105	4990
8	5040	5115	5050	5090	5165	5260	5278	5250	5250	5160	5090	5000
9	5000	5110	5060	5080	5160	5250	5273	5255	5273	5180	5115	5000
10	4950	5150	5090	5110	5200	5260	5261	5255	5238	5180	5100	4979
11	5080	5135	5070	5140	5210	5250	5258	5255	5245	5170	5080	5045
12	5070	5150	5080	5140	5190	5250	5262	5255	5243	5163	5118	5011
13	5060	5130	5090	5120	5230	5250	5260	5255	5250	5165	5075	5012
14	4900	5110	5100	5110	5180	5210	5280	5240	5253	5165	5115	5043
15	4930	5080	5110	5100	5180	5235	5291	5263	5205	5174	5059	5007
16	5080	5160	5120	5100	5190	5265	5290	5241	5256	5158	5124	5006
17	5070	5105	5100	5080	5210	5260	5292	5245	5250	5161	5109	4983
18	5065	5100	5130	5070	5195	5250	5292	5259	5247	5138	5011	4968
19	5070	5100	5110	5090	5220	5215	5281	5264	5239	5165	5058	4988
20	5065	5105	5080	5090	5240	5230	5292	5260	5230	5130	5059	4960
21	5080	5110	5140	5090	5230	5265	5306	5255	5240	5135	5050	5040
22	5085	5115	5080	5110	5240	5250	5300	5270	5212	5128	5032	5015
23	5060	5100	5090	5110	5250	5265	5284	5275	5190	5155	5045	5010
24	5060	5115	5090	5105	5250	5250	5291	5278	5182	5163	5021	5008
25	5080	5125	5080	5100	5250	5265	5281	5252	5144	5130	5005	5040
26	5070	5090	5125	5100	5260	5260	5289	5270	5121	5109	5065	5032
27	5140	5110	5135	5115	5265	5280	5281	5280	5146	5125	5062	5050
28	5140	5095	5140	5110	5265	5275	5281	5280	5148	5125	5011	4961
29	5090		5110	5115	5270	5270	5286	5260	5123	5097	5005	5050
30	5150		5110	5140	5275	5250	5285	5258	5130	5120	5011	5000
31	5090		5125		5270		5285	5272		5048		5040
Mean	5069	5118	5096	5108	5205	5257	5276	5263	5221	5138	5068	5011

SOME STUDIES ON VOLCANO ASO AND KUJIU (PART 15). ON THE EXPLOSION OF VOLCANO ASO IN 1958

Munetosi NAMBA

(received on Jan. 1st, 1959)

Abstract

From the analysis of the tree growth curve the writer expected for 1906 type explosion of Aso-crater. On 19 th June 1958 severe earthquakes occurred in Ariake sea area (about 40 km West from Aso), then he expected for Aso explosion, too, because in active stage of Aso volcano it often explodes after a severe nearby earthquake, and more it was just in a dry season after an extra ordinal rainy season. It is in this case in June of 1958.

At about 15^{m} 22^{h} 24^{th} June 1958 (that is, after the hour of maximum atmospheric pessure) the first crater of Volcano Aso took a severe explosion. Considerable damages were done to the institutes around the crater. This writer analysed the gusts into four groups and calculated their energies. And he discussed the tilting of the ground near the crater, too, and found out that the north component extraordinally rose before the explosion occurrs.

1. Forewords.

From the analysis of the tree growth curve the writer expected for 1906 type explosion of Aso crater.[1] On 19th. June 1958 severe earthquakes occurred at ARIAKE sea area (about 40 km West from Aso), then he expected, too, for an Aso explosion, because in active stage of Aso Volcano, it often explodes after a severe nearby earthquake. And more it was just in a dry season after an extra ordinal rainy season. It was the case in June of 1958. At about 15^{m} 22^{h} 24^{th} June 1958 (that is, after the hour of maximum atmospheric pressure) the first crater of Aso Volcano took a severe explosion.[2]

Considerable demages were done to the institutes around the crater. This writer analysed the gusts by the explosion into four groups and calculated their energies. And then he discussed the tilting of the ground near the cratter, and found out that the norh component extraordinally rose before the explosion occurrs.

2.

Microtremors appeared on the records (600 times) at Aso weather station after 21^{h} 15^{m} 24^{th} June 1958, it became maximum in its amplitude at about 21^{h} 26^{m}, and it disappeared after 21^{h} 36^{m}. At about 22^{h} 13^{m} 13^{sec} the first explosion took place at the first crater. At about 22^{h} 16^{m} 30^{sec} the amplitude became maximum, that is that the boring stage ended and next it transfered into its magma stage. At about 22^{h} 17^{m} the

magma stage became to its end, and the crater regained its stillness. The boring stage acted through about 1^{m} 38^{sec}, ejecters were mere rock ashes and rock bombs. The magma stage acted during about 40^{sec}, ejecters are mere Volcanic gas. Total ashes were about 8×10^{5} tons, and total energy is about 3.1×10^{19} Erg.

3.

Distriburions of Volcanic ashes are figured in Fig. 1.

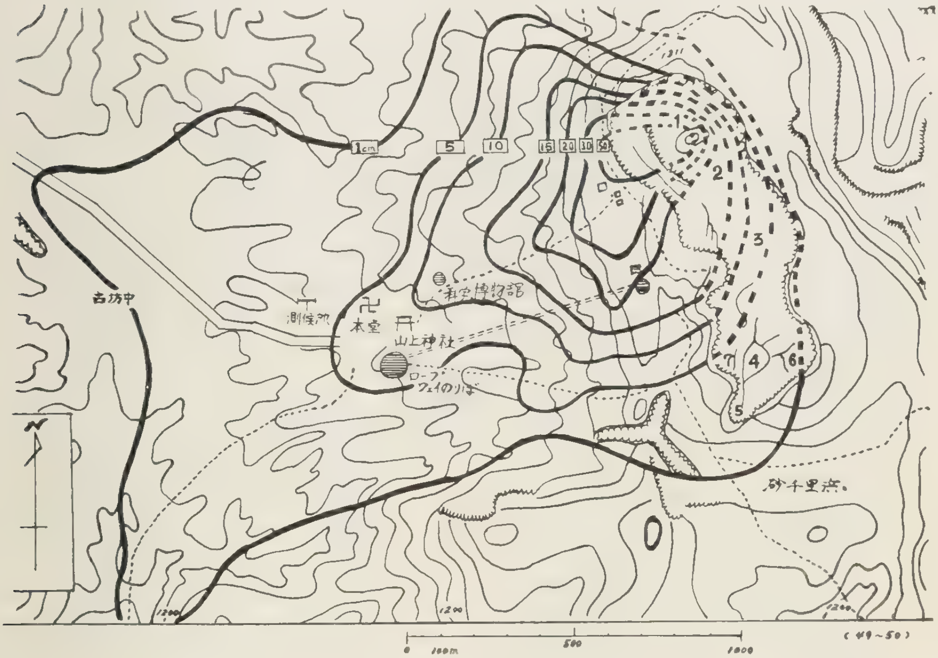


Fig. 1. Distribution of Ashes (1959 June 24th 22^{h})

Meteorological data at Aso weather station are as follows:-
The station's departure from the first crater:- 1200 meters
The atmospheric temperature was:- 16°C (22h)
The atmospheric pressure was:- 878.2 mb.
The wind velocity was:- NE 11.5 m/sec.
Maximum excess pressure by the detonation tone was:- 4.9 mb.
Altitude of the station is about 1000 meters above the sea.
In this paper the writer assumes as follows:
Departure of Crater-station from the crater is 450 meter;
Departure of the fourth rope-way pillar from the crater is 450 meter;
Departure of Volcanic-museum from the crater is 1000 meter;
Departure of Aso weather station from the crater is 1200 meter.

4. Velosity of Gusts.

From the top of the fourth ropeway pillar (whose hight is 12.5 meters and its departure from the crater is about 450 meters) a repair man (mr. Takumi Oyama) was blown down to SW direction by the first gust. The fell down position was 12.5 meters SW from the pillar.

Then its horizontal velocity must be 8 m/sec, and then if the velocity of the gust is V m/sec,

$$V=100/\sqrt{\rho}\ \text{m/sec}$$

in which the weight of the body is about 50 kgr, the oppressed area is 0.5 square meters, the density of the atmosphere is ρ kgr/cubic meters, the effective duration is about 1/10 sec, and the Reynold's constant C is 1/10.

Now the density of the atmophere was 1.17 kgr/cubic meters as showed in the following chapter. Then the velocity of the first gust must be ${}_{450}V_2=92$m/sec.

A signal pole near Aso weather station (about 1200 meters SW from the crater) was bent about 20 degrees by the gust (fig. 3).

The velosity of the first gust was calculated as about ${}_{1200}V_1=52$ m/sec, when ρ is 1.100 kgr/cubic meter.

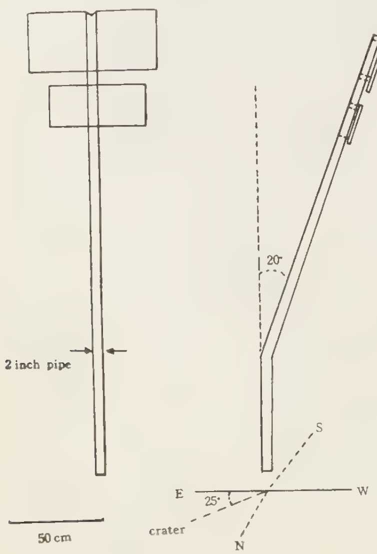


Fig. 3 Signal pole near Aso weather station

Fig. 2. The tourth ropeway pole (Departure 450m SW)

5.

Observed depth of volcanic ashes over the damage area is tabulated in Table (1).

From the above data the following experimental formula is reduced as

Table 1.

Depth of ashes (after ASO weather station)

x (distance from the crater)	H (mean depth)
100 m	46 cm
190 m	30 cm
260 m	20 cm
450 m	10 cm
900 m	2 cm
1200 m	1 cm

$$H=56\times e^{-8.36\times10^{-3}\cdot x}$$

in which H:–in cm (mean depth of ashes at x meters from the crater),

x:–in meters (departure from the crater).

Calculated depth at x meters are tabulated in Table (2).

Table 2.

x meter	Calc. H cm	x meter	Calc. H cm
0	56	800	4
100	40	900	3
200	29	1000	2
300	21	1100	1
400	15	1200	1
450	12	1300	0.7
500	11	1400	0.5
600	8	1500	0.4
700	5	—	
		1900	0.1

Fig. 4. Plasted yona on the North side wall of the crater station

Volcanic ashes (yona) from the crater was plasted in four layers on the north side wall of the crater station as Fig. (4).

The thickness of the first layer is 3.5 cm, that of the second layer is 2.5 cm, that of the third layer was 1.5 cm, and of the fourth layer is 0.5 cm. Therefore their ratios are

7/16: 5/16: 3/16: 1/16

and the energies of the explosions can be divided into four groups as Fig. (5) too.

That is:– For 1^{st}. explosion:
925/4532, duration 33^{sec},
For 2^{nd} explosion:
1415/4532, duration 40^{sec},
For 3^{rd} explosion:
1540/4532, duration 20^{sec},
For 4^{th} explosion:– 652/4532, duration 5^{sec}.

and total duration is 98^{sec}. At Aso weather station (its departure about 1200 m) four groups are recorded in the barogram by the detonation tones, too. (Fig. 7)
From the above data the depth of ashes was defined as

$$H = 56e^{-3.6\times10^{-3}\cdot x}$$

Fig. 4. Section of plasted yona

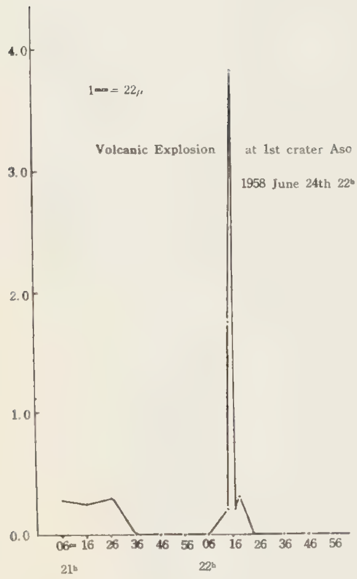


Fig. 5. A. Amplitude of volcanic micro tremors

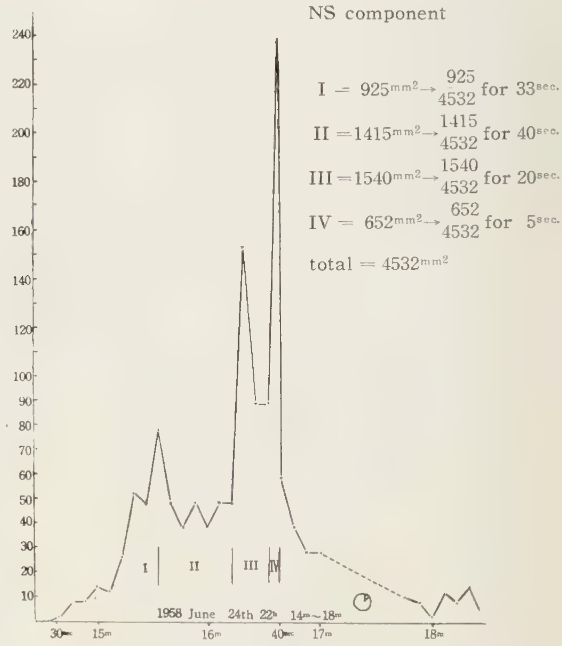


Fig. 5. B. Energy of explosion for every group

Therefore total amount of ashes between 450 meters and infinity must be

$$S_{450} = \int_{450}^{\infty} H\cdot dx = \frac{12}{3.36\times10^{-3}} = 3582^{cm} \doteqdot 36^{m}.$$

If the hight of the gust is about 50 meters, the amount of ashes must be 71.6 cm for a meter of atmosphere. As the density of ash is 1.5 the weight of ashes for a cubic meter of the atmosphere must be $5.5059\times7/16$ that is 0.2409 kgr was carried during 33 seconds, that is that 0.073 kgr was carried per second.
Therefore mean density of the first gust must be

$$_{450}\rho_1 = 1.10 + 0.073 = 1.17\ \text{kgr/m}^3,$$

and

for the 2^{nd} gust ${}_{450}\rho_2=1.10+0.043 \ =1.14$ kgr/m^3,

for the 3^{rd} gust ${}_{450}\rho_3=1.10+0.05124=1.15$ kgr/m$_3$,

for the 4^{th} gust ${}_{450}\rho_4=1.10+0.0688 \ =1.17$ kgr/m^3,

Treating by the same process as above, at Aso weather station (1200^m from the crater) the density of the gust are calculated as follows:-

for the 1^{st} gust ${}_{1200}\rho_1=1.10+0.0042=1.104$ kgr/m^3,

for the 2^{nd} gust ${}_{1200}\rho_2=1.10+0.002 \ =1.102$ kgr/m^3,

for the 3^{rd} gust ${}_{1200}\rho_3=1.10+0.003 \ =1.103$ kgr/m^3,

for the 4^{th} gust ${}_{1200}\rho_4=1.10+0.004 \ =1.104$ kgr/m^3,

Then it is said that the density of the gust at 1200 meters was the same value of the air.

By this reason the velocity of the gust at 1200 meters must be as follows:-

the velocity of the 1^{st} gust ${}_{1200}V_1 \doteqdot 52$m/sec for 33^{sec}, ${}_{1200}\rho_1=1.104$ kgr/m^3,

the velocity of the 2^{nd} gust ${}_{1200}V_2 \doteqdot 52$m/sec for 40^{sec}, ${}_{1200}\rho_2=1.102$ kgr/m^3,

the velocity of the 3^{rd} gust ${}_{1200}V_3 \doteqdot 52$m/sec for 20^{sec}, ${}_{1200}\rho_3=1.103$ kgr/m^3,

the velocity of the 4^{th} gust ${}_{1200}V_4 \doteqdot 52$m/sec for 5^{sec}, ${}_{1200}\rho_4=1.104$ kgr/m^3.

Now at Crater Station (departure about 450^m), ${}_{450}\rho_1=1.17$kgr/m^3, therefore the velocity of the 1^{st} gust ${}_{450}V_1=\dfrac{10^2}{\sqrt{1.17}} \doteqdot 92$m/sec,

The velocity of gust must damp as $V=V_0\,e^{-\alpha x}$

where

V_0:- velocity of gust at the crater.

α :- damping coefficient,

x :- departure from the crater,

V :- velocity of the gust at x meters.

${}_{1200}V_1$:- 52m/sec at $x=1200^m$,

${}_{450}V_1$:- 92m/sec at $x=\ 450^m$.

Therefore we have

$$V_1=130\times e^{-7.6\times10^{-4}\cdot x}$$

and then the velocity of gust must be ${}_0V_1=130$m/sec, but at that time as the wind velocity was 12 m/sec, the proper velocity of gust by the explosion must be ${}_0V_{1G}=118$ m/sec.

By the same process total amount of ash at the origine must be calculated as 166.70 meters. Then 37.933 kgr of ashes must be contained in a cubic meters of the gust. And therefore

the mean density of the 1^{st} gust ${}_0\rho_1=1.10+0.503=1.60$ kgr/m^3,

the mean density of the 2^{nd} gust ${}_0\rho_2=1.10+0.296=1.396$kgr/m^3,

the mean density of the 3rd gust ${}_0\rho_3=1.10+0.356=1.456\text{kgr/m}^3$,
the mean density of the 4th gust ${}_0\rho_4=1.10+0.474=1.574\text{kgr/m}^3$.
By the equation ${}_xV_1=130\ e^{-7.6\times10^{-4}\cdot x}$ the distance x at which
${}_xV_1=12\text{m/sec}$ is calculated as $x=3133^{\text{m}}\doteqdot3^{\text{k.m.}}$

That is that the gust must vanish at about 3 km (SW) from the crater.

6. The Energy of the Explosion.

Now by the first explosion

Vertical sectin of the gust: $A=50\times1000=5\times10^4\ \text{m}^2$,
Density of the gust:– ${}_0\rho_1=1.60\ \text{kgr/m}^3$,
Velocity of the gust:– ${}_0V_1=130\ \text{m/sec}$, ${}_0V_{G1}=118\text{m/sec}$,
Duration of the explosion:– $t=33$ sec.

Therefore we have the emitted mass M_1 as

$$M_1={}_0\rho_1\times{}_0V_1\times A\times t=1.60\times130\times5\times10^4\times33$$
$$=3.432\times10^8\text{kgr}.$$

Then total mass of ash emitted through the explosion must be

$$M=3.432\times10^8\times16/7=7.85\times10^8\text{kgr}$$
$$\doteqdot8\times10^{5}\text{ton}$$

And the energy E_1 of the first gust must be

$$E_1=1/2M_1\ {}_0V_{G1}^2=1/2\times3.432\times18^8\times10^3\times(118\times10^2)^2$$
$$=2.0\times10^{19}\ \text{erg}$$

and M_2 of the 2nd gust

$$M_2={}_0\rho_2\times{}_0V_2\times A\times t=M\times\frac{5}{10}=2.45\times10^8\text{kgr}$$

in which

$$M=7.85\times10^8\text{kgr}$$
$${}_0\rho_2=1.40\text{kgr/m}^3$$
$$A=50\times1000\text{m}^2$$
$$t=40^{\text{sec}}$$
$$\therefore\ {}_0V_2=88\text{m/sec}$$
$$\therefore\ {}_0V_{G2}=76\text{m/sec}$$
$$\therefore\ E_2=\frac{1}{2}M_2\ {}_0V_{G2}^2\doteqdot0.1\times10^{19}\text{Erg}.$$
$$M_3={}_0\rho_3\times{}_0V_3\times A\times t=M\times3/16=1.47\times10^8\text{kgr}.$$

where

$${}_0\rho_3=1.456\text{kgr/m}^3$$

$$A = 50\times 1000\text{m}^2$$
$$t = 20\text{sec}$$
$$\therefore \quad V_3 = 101\text{m/sec}$$
$$\therefore \quad {}_0V_{G3} = 101-12=89\text{m/sec}$$
$$\therefore \quad E_3 = \frac{1}{2} M_3\times {}_0V_{G3}^2 \fallingdotseq 0.65\times 10^{19}\text{Erg.}$$

then

$$M_4 = 4.9\times 10^7\text{kgr}$$
$${}_0V_4 = 125\text{m/sec}$$
$${}_0V_{G4} = 113\text{m/sec}$$
$$E_4 = 0.3\times 10^{19}\text{Erg}$$

Therefore atlast we have the total energy E as

$$E=E_1+E_2+E_3+E_4=3.1\times 10^{19}\text{Erg.}$$

Apparent velocity of Gust		Original velocity of Gust		Density of Gust		Emitted mass		Eenergy	
	m/sec		m/sec		kgr/m^3		$\times 10^8$kgr		$\times 10^{19}$Erg
${}_0V_1$	130	${}_0V_{G1}$	118	${}_0\rho_1$	1.60	${}_0M_1$	3.432	E_1	2.0
${}_0V_2$	88	${}_0V_{G2}$	76	${}_0\rho_2$	1.396	${}_0M_2$	2.45	E_2	0.1
${}_0V_3$	101	${}_0V_{G3}$	89	${}_0\rho_3$	1.456	${}_0M_3$	1.47	E_3	0.65
${}_0V_4$	125	${}_0V_{G4}$	113	${}_0\rho_4$	1.574	${}_0M_4$	0.49	E_4	3.1

Duration	
	sec
t_1	33
t_2	40
t_3	20
t_4	5

If we can assume that the damping coefficient of the gust keeps the same value through the explosion,

$$V_2 = 88e^{-[\text{illegible}]\cdot x}$$
$$V_3 = 101e^{-[\text{illegible}]\cdot x}$$
$$V_4 = 113e^{-7.6\times 10^{-4}\cdot x},$$

Therefore we have the following table.

app. velocity of Gust \ distance	$x=450^{\text{m}}$ (Crater station)	$x=900^{\text{m}}$ (Volcano Museum)	$x=1200^{\text{m}}$ (Aso weather station)	density \ distance	$x=450^{\text{m}}$	$x=900^{\text{m}}$	$x=1200^{\text{m}}$
	m/sec	m/sec	m/sec		kgr/m^3		kgr/m^3
V_1	92	61	52	ρ_1	1.17		1.104
V_2	63	41	35	ρ_2	1.14		1.102
V_3	72	47	41	ρ_3	1.15		1.103
V_4	80	53	45	ρ_4	1.16		1.104

7. On the Detonation Tone.

At the time of the explosion, the atmospheric temperature was 16°C, and the atmospheric pressure was 878.2 mb. at Aso Weather Station.
If the meteorological data in and around the crater were equall to that of the station, the density of the atmosphere around the crater must be 1.1×10^{-3}, then the velocity of the detonation tone (C) must be three times of that of the gust.

$$C=\sqrt{\frac{\chi'}{\rho}}=335.26\text{m/sec.}$$

By the barogram at the weather station we can assume that four groups of explosion were occurred in this time as

No. 1 explosion ······ 4.9 mb.,
No. 2 explosion ······ 3.5 mb.,
No. 3 explosion ······ 2.1 mb.,
No. 4 explosion ······ 0.8 mb..

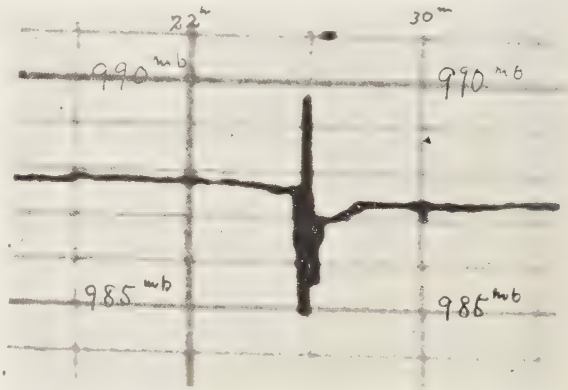


Fig. 6. Barogram by detonation tone (at Aso weather station)

8. On the Tilting.

Relative tilting at Aso Weather Station during one year (1957 July to 1958 July) is tabulated as Table 3, and calculated most probable values are reduced as follows:-
EW-component:-

$$\begin{aligned}&8.''58\cos(t-7)\\+\;&4.''14\cos(2t-213)\\+\;&4.''31\cos(3t-17)\\+\;&1.''12\cos(4t-194)\\+\;&1.''17\cos(6t-126)\end{aligned}$$

(in which positive sighn means W-ward up)
NS-component:-

$$\begin{aligned}&8.''20\cos(t-58)\\+\;&12.''50\cos(2t-95)\\+\;&2.''15\cos(3t-286)\\+\;&9.''63\cos(4t-150)\\+\;&5.''73\cos(6t-156)\end{aligned}$$

(in which positive sighn means N-ward up).
These are graghed as Fig. 7, and their vector diagrams are showed as Fig. 8.
Viewing the results, we can find that the tilting at Aso is predominated in NS-component than EW-component. This is the same as that of microtremors, and these

relations are the same as that at Kyoto Volcanic Laboratory. The explosion of Aso crater (1957 Dec. and 1956 June) may be said that after the maximum upward tilting of N-ward the explosions occurred.

Table 3A

EW [W+]

	N_1	N_2	N_3	N_4	N_6	t
	″	″	″	″	″	
Jan.	8.53	$\bar{3}.47$	4.12	$\bar{1}.09$	$\bar{0}.69$	0
	7.96	$\bar{1}.88$	2.02	$\bar{0}.31$	$\bar{0}.95$	1
Feb.	6.85	0.22	$\bar{1}.26$	0.78	0.69	2
	5.28	2.26	$\bar{3}.81$	1.09	0.95	3
Mar.	3.35	3.69	$\bar{4}.12$	0.31	$\bar{0}.69$	4
	1.19	4.13	$\bar{2}.02$	$\bar{0}.78$	$\bar{0}.95$	5
Apr.	$\bar{1}.50$	3.47	1.26	$\bar{1}.09$	0.69	6
	$\bar{3}.21$	1.88	3.81	$\bar{0}.31$	0.95	7
May	$\bar{5}.16$	$\bar{0}.22$	4.12	0.78	$\bar{0}.69$	8
	$\bar{6}.76$	$\bar{2}.26$	2.02	1.09	$\bar{0}.95$	9
June	$\bar{7}.90$	$\bar{3}.69$	$\bar{1}.26$	0.31	0.69	10
	$\bar{8}.50$	$\bar{4}.13$	$\bar{3}.81$	$\bar{0}.78$	0.95	11
July	$\bar{8}.53$	$\bar{3}.47$	$\bar{4}.12$	$\bar{1}.09$	$\bar{0}.69$	12
	$\bar{7}.96$	$\bar{1}.88$	$\bar{2}.02$	$\bar{0}.31$	$\bar{0}.95$	13
Aug.	$\bar{6}.85$	0.22	1.26	0.78	0.69	14
	$\bar{5}.28$	2.26	3.81	1.09	0.95	15
Sept.	$\bar{3}.35$	3.69	4.12	0.31	$\bar{0}.69$	16
	$\bar{1}.19$	4.13	2.02	0.78	$\bar{0}.95$	17
Oct.	1.05	3.47	$\bar{1}.26$	$\bar{1}.09$	0.69	18
	3.21	1.88	3.81	$\bar{0}.31$	0.95	19
Nov.	5.16	$\bar{0}.22$	$\bar{4}.12$	0.78	0.69	20
	6.76	$\bar{2}.26$	$\bar{2}.02$	1.09	0.95	21
Dec.	7.90	$\bar{3}.69$	1.26	0.31	0.69	22
	8.50	$\bar{4}.13$	3.81	$\bar{0}.78$	0.95	23

Table 3B

EW [W+]

	N_1+N_2	$N_3+N_4+N_6$	Total	Observed Values
	″	″	″	″
Jan.	5.05	2.34	7.39	9.26
	6.08	0.76	6.84	13.39
Feb.	7.07	0.21	7.28	9.93
	7.54	$\bar{1}.77$	5.77	11.50
Mar.	7.04	$\bar{4}.50$	2.54	10.95
	5.32	$\bar{3}.75$	1.57	7.55
Apr.	2.42	0.86	3.28	6.63
	$\bar{1}.33$	4.45	3.12	8.74
May	5.38	4.21	$\bar{1}.17$	6.33
	$\bar{9}.02$	2.15	$\bar{6}.87$	$\bar{1}.97$
June	$\overline{11}.59$	$\bar{0}.26$	$\overline{11}.85$	$\bar{7}.86$
	$\overline{12}.63$	$\bar{3}.64$	$\overline{16}.27$	$\bar{7}.20$
July	$\overline{12}.00$	$\bar{5}.30$	$\overline{17}.30$	$\bar{7}.94$
	9.84	$\bar{3}.28$	$\overline{13}.14$	$\bar{9}.74$
Aug.	$\bar{6}.63$	2.73	3.90	6.20
	$\bar{3}.02$	5.85	3.83	6.41
Sept.	0.34	3.74	4.08	7.67
	2.94	0.29	3.23	10.40
Oct.	4.52	$\bar{1}.66$	2.86	12.45
	5.09	$\bar{3}.17$	1.92	5.60
Nov.	4.94	4.03	0.91	6.13
	4.50	$\bar{1}.88$	2.62	7.96
Dec.	4.20	2.26	6.46	13.30
	4.37	3.96	8.33	13.98

$$5.82''+8.58''\cos(t-7^\circ)+4.14\cos(2t-213^\circ)+4.31\cos(3t-17^\circ)$$
$$+1.12\cos(4t-194^\circ)+1.17\cos(6t-126^\circ)$$

Table 3C

NS [N+]

	N_1	N_2	N_3	N_4	N_6	t
Jan.	4.35″	$\bar{1}.09$″	0.59″	$\bar{8}.34$″	$\bar{4}.91$″	0
	6.00	$\bar{7}.19$	1.88	$\bar{8}.34$	$\bar{2}.18$	1
Feb.	7.24	$\overline{11}.33$	2.07	0.00	4.91	2
	8.00	$\overline{12}.45$	1.04	8.34	2.18	3
Mar.	8.20	$\overline{10}.24$	$\bar{0}.59$	8.34	$\bar{4}.91$	4
	7.84	$\bar{5}.28$	$\bar{1}.88$	0.00	$\bar{2}.18$	5
Apr.	6.95	1.09	$\bar{2}.07$	$\bar{8}.34$	4.91	6
	5.59	7.17	$\bar{1}.04$	$\bar{8}.34$	2.18	7
May	3.85	11.33	0.59	0.00	$\bar{4}.91$	8
	1.85	12.45	1.88	8.34	$\bar{2}.18$	9
June	$\bar{0}.29$	10.24	2.07	8.34	4.91	10
	$\bar{2}.30$	5.28	1.04	0.00	2.18	11
July	$\bar{4}.35$	$\bar{1}.09$	$\bar{0}.59$	$\bar{8}.34$	$\bar{4}.91$	12
	$\bar{6}.00$	$\bar{7}.17$	$\bar{1}.88$	$\bar{8}.34$	$\bar{2}.18$	13
Aug.	$\bar{7}.24$	$\overline{11}.33$	$\bar{2}.07$	0.00	4.91	14
	$\bar{8}.00$	$\overline{12}.45$	1.04	8.34	2.18	15
Sept.	$\bar{8}.20$	$\overline{10}.24$	0.59	8.34	$\bar{4}.91$	16
	$\bar{7}.84$	$\bar{5}.28$	1.88	0.00	$\bar{2}.18$	17
Oct.	$\bar{6}.95$	1.09	2.07	$\bar{8}.34$	4.91	18
	$\bar{5}.59$	7.17	1.04	$\bar{8}.34$	2.18	19
Nov.	$\bar{3}.85$	11.33	$\bar{0}.59$	0.00	$\bar{4}.91$	20
	$\bar{1}.85$	12.45	$\bar{1}.88$	8.34	$\bar{2}.18$	21
Dec.	0.29	10.24	$\bar{2}.07$	8.34	4.91	22
	2.40	5.28	$\bar{1}.04$	0.00	2.18	23

Table 3D

NS [N+]

	N_1+N_2	$N_3+N_4+N_6$	Total	Observed Values	t
Jan.	3.26″	$\overline{12}.66$″	$\bar{9}.40$″	$\bar{9}.39$″	0
	$\bar{1}.19$	$\bar{8}.64$	$\bar{9}.83$	$\bar{0}.29$	1
Feb.	$\bar{4}.09$	6.98	2.89	9.25	2
	$\bar{4}.45$	11.66	7.21	9.40	3
Mar.	$\bar{2}.04$	2.84	0.80	3.06	4
	2.56	$\bar{4}.06$	$\bar{1}.50$	1.86	5
Apr.	8.04	$\bar{5}.50$	2.54	9.19	6
	12.76	$\bar{7}.20$	5.56	12.40	7
May	15.18	$\bar{4}.30$	10.88	18.06	8
	14.30	8.04	22.34	21.00	9
June	9.95	15.32	25.27	22.64	10
	2.98	3.22	6.20	24.13	11
July	$\bar{5}.44$	$\overline{13}.84$	$\overline{19}.28$	$\overline{20}.86$	12
	$\overline{13}.17$	$\overline{12}.40$	$\overline{25}.57$	$\overline{17}.14$	13
Aug.	$\overline{18}.57$	2.84	$\overline{15}.73$	15.93	14
	$\overline{20}.45$	9.48	$\overline{10}.97$	$\bar{9}.21$	15
Sept.	$\overline{18}.44$	4.02	$\overline{14}.42$	$\bar{6}.57$	16
	$\overline{13}.12$	0.30	$\overline{12}.82$	$\bar{5}.43$	17
Oct.	$\bar{5}.86$	1.36	$\bar{4}.50$	$\bar{5}.10$	18
	1.58	$\bar{5}.12$	$\bar{3}.54$	$\bar{1}.37$	19
Nov.	7.48	$\bar{5}.50$	1.98	7.27	20
	10.60	4.26	14.86	19.03	21
Dec.	10.53	11.18	21.71	30.36	22
	7.68	1.14	8.82	10.00	23

$$4.43'' + 8.20'' \cos(t-58^\circ) + 12.50'' \cos(2t-95^\circ) + 2.15'' \cos(3t-286^\circ)$$
$$+ 9.63'' \cos(4t-150^\circ) + 5.73'' \cos(6t-156^\circ)$$

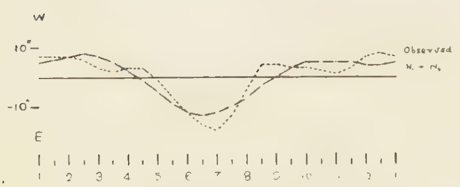


Fig. 7A WE-Component of tilting

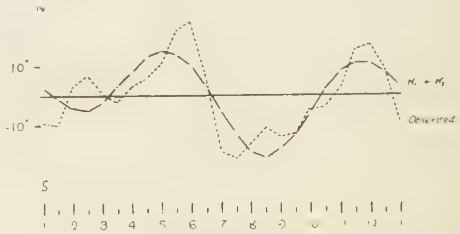


Fig. 7B NS-Component of tilting

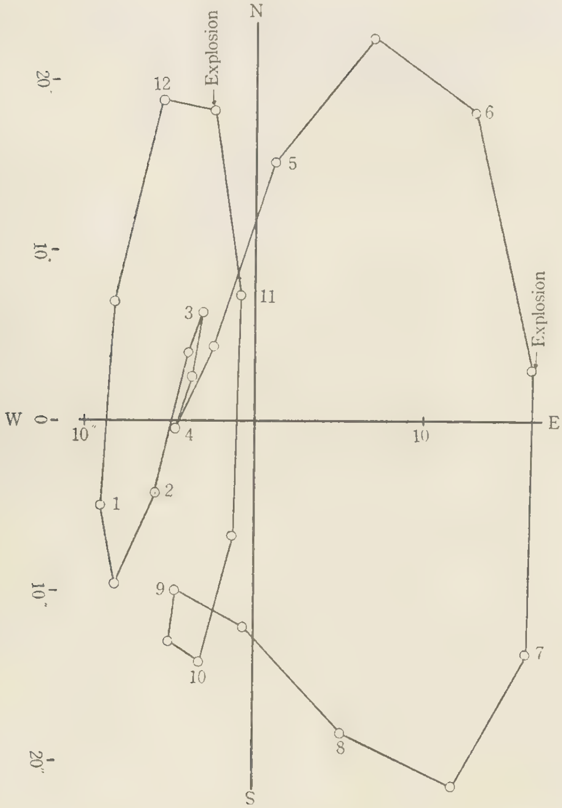


Fig. 8 Tilting vector diagram at Aso

References

1) Read at the meeting of Japan Physical Society (Kyoto, 1945)
2) 南葉: —火山爆発に及ぼす気圧変化の影響; —地球物理, Vol. 6, No. 3, 1942, Dec.

THE EFFECT OF THE INTERNAL STRESS ON THE LONGITUDINAL MAGNETOSTRICTION IN WEAK MAGNETIC FIELD (III)

Shigeo MATSUMAE

(Received March 31, 1959)

〔I〕 Introduction

The present writer studied the longitudinal magnetostriction of Ni in weak magnetic field.[(1)(2)(3)] The longitudinal magnetostriction is frequently found even in the very weak magnetic field. This is caused probably by the unstable domain or domain-boundary under the suitable internal strain in the specimen.

We study the longitudinal magnetostriction in Ni, Fe, Fe-Ni alloys and others under various degree of external stress. The experimental results are as follows.

〔II〕 Experimental Results

The specimens are annealed at 900°C for 2 hours and then cooled to the room-temperature with the furnace.

The specimens are, at first, demagnetized with the alternate magnetic field having the maximum amplitude of about 6 Oersted. After demagnetizing, the tensive or compressive stress is applied to the specimen, according to the positive or negative longitudinal magnetostriction.

Apparatus of the measurement and experimental method are the same as described in the previous papers.[(1)(2)(3)]

The experimental results are as follows.

(1) Nickel

The specimen is a round bar of nickel, containning a small amount of manganese, 2.5 mm in diameter and 19 cm in length. The specimen is inserted in the brass cylinder, of which diameter is almost equal to that of the specimen, in order to prevent the specimen from bending, and then a compressive external stress is applied to the specimen. The experimental results are shown in Fig. 1 and Fig. 1–A.

As shown in the figures, the longitudinal magnetostriction is observed in the weak magnetic field less than about 0.3 Oersted and shows a maximum under the compressive external stress of about 0.2 kg/mm^2 and then decreases with the increase of a compressive external stress. Such a tendency which the longitudinal magnetostriction shows a maximum in weak magnetic field under a suitable external stress, is generally observed in other ferromagnetic metals and alloys.

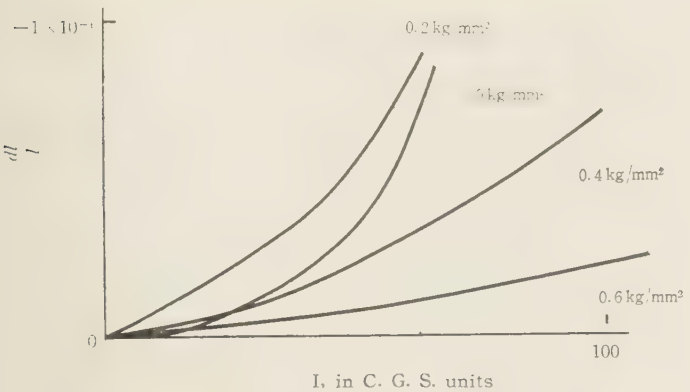


Fig. 1 The longitudinal magnetostriction of Ni under various degrees of tension, when demagnetizing is applied at first and compression subsequently.

(2) Fe-Ni (69.93%) alloy

The specimen is a round bar, 1.5 mm in diameter and 21 cm in length. The experimental results are shown in Fig. 2 and Fig. 2-A.

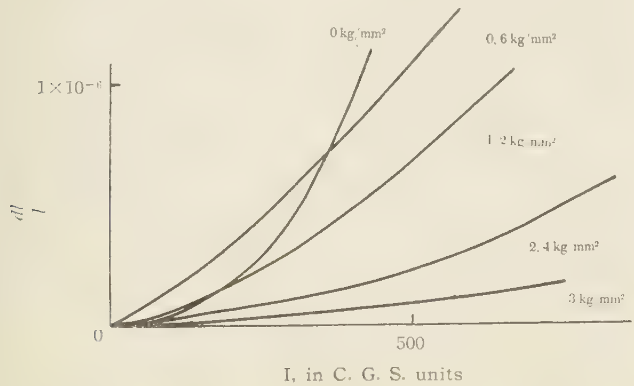


Fig. 2 The longitudinal magnetostriction of Fe-Ni (69.93%) alloy under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

(3) Fe-Ni (50.24%) alloy

The specimen is a round bar, 1.5 mm in diameter and 21 cm in length. The experimental results are shown in Fig. 3 and Fig. 3-A.

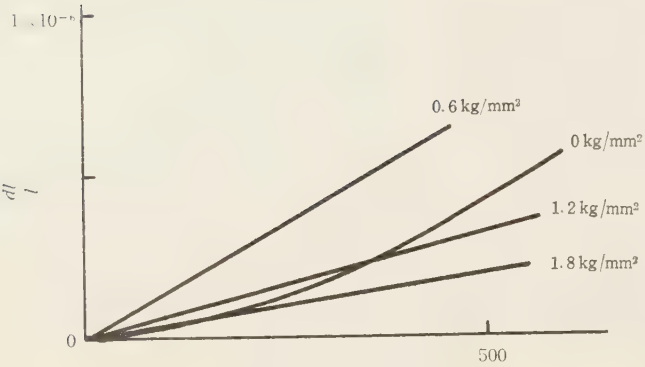


Fig. 3 The longitudinal magnetostriction of Fe-Ni (50.24%) alloy under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

(4) Fe-Ni (20.87%) alloy

The specimen is a round bar, 2 mm in diameter and 21 cm in length. The experimental results are shown in Fig. 4 and Fig. 4-A.

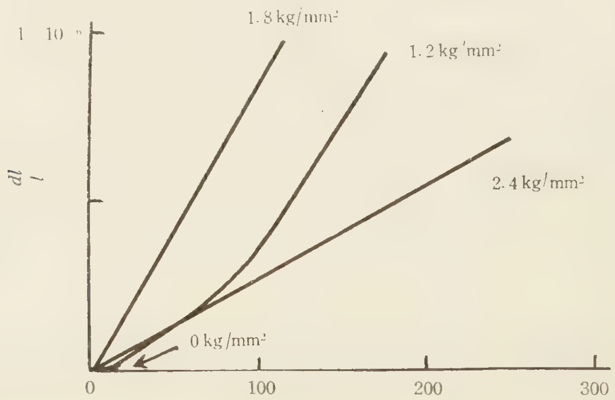


Fig. 4 The longitudinal magnetostriction of Fe-Ni (20.87%) alloy under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

(5) Fe

The specimen is a round bar, 1.5 mm in diameter and 21 cm in length. The experimental results are shown in Fig. 5 and Fig. 5-A.

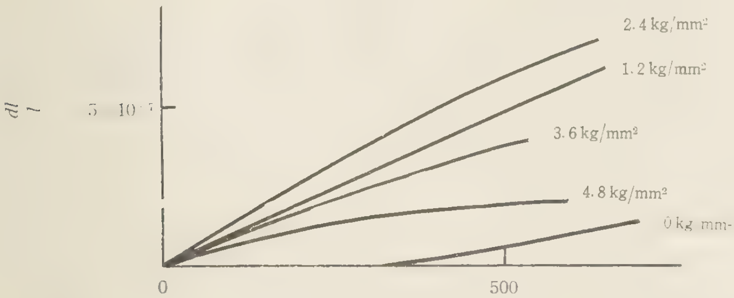


Fig. 5 The longitudinal magnetostriction of Fe alloy under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

(6) Fe–Co (19.59%) alloy

The specimen is a round bar, 2 mm in diameter and 21 cm in length. The experimental results are shown in Fig. 6 and Fig. 6–A. (This specimen is not linear)

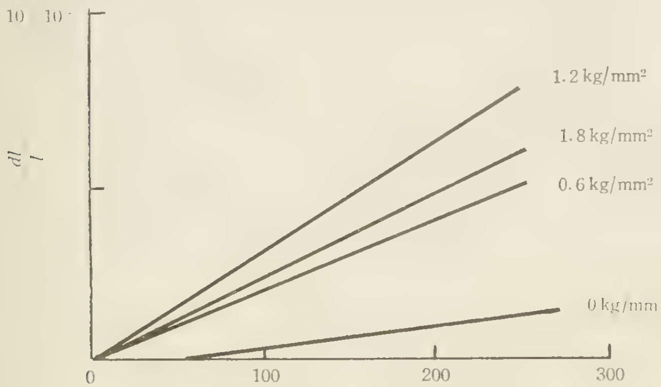
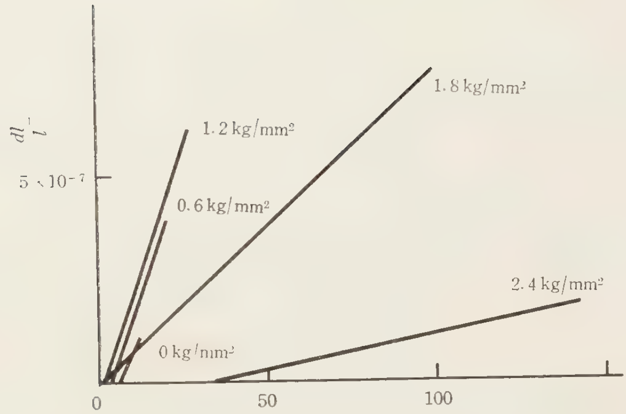


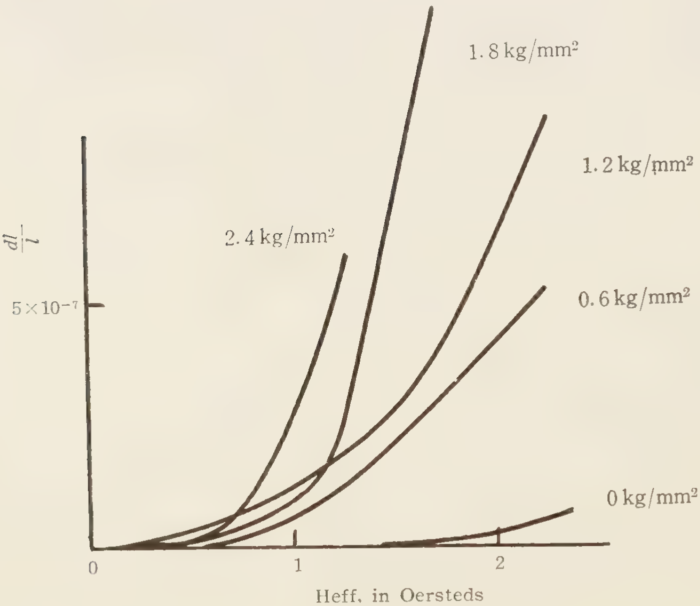
Fig. 6 The longitudinal magnetostriction of Fe–Co (19.59%) alloy under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

(7) Fe–Co (68.85%) alloy

The specimen is a round bar, 2 mm in diameter and 21 cm in length. The experimental results are shown in Fig. 7 and Fig. 7–A. (This specimen is not linear)



I vs $\frac{dl}{l}$ curves



$\frac{dl}{l}$ — Heff curves

Fig. 7 The longitudinal magnetostriction of Fe–Co (68.85%) alloy under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

(8) **Co**

The specimen is a round bar, 2.5 mm in diameter and 21 cm in length.
The experimental results are shown in Fig. 8–A.

As shown in the figures, the longitudinal magnetostriction is almost unobservable.

In the weak magnetic field, the longitudinal magnetostriction increases at first with the increase of the external stress and reachs a maximum under a suitable degree of the external stress and then decreases with the increase of the external stress. This tendency seems to be observed generally in ferromagnetic metals and alloys.

References

(1) S. Matsumae: "The Effect of the Heat-treatment on the Longitudinal Magnetostriction in Weak Magnetic Field." Kumamoto Journal of Science Vol. 2 No. 2 1955.

(2) S. Matsumae: "The Effect of the Internal Stress on the Longitudinal Magnetostriction in Weak Magnetic Field." Kumamoto Journal of Science Vol. 2 No. 2 1955.

(3) S. Matsumae: "The Effect of the Internal Stress on the Longitudinal Magnetostriction in Weak Magnetic Field (II)." Kumamoto Journal of Science Vol. 2 No. 3 1955.

Correction

In the previous paper (The Longitudinal Magnetostriction of Fe–Ni Alloys and Fe–Co Alloys in Weak Magnetic Field), the magnitude of the longitudinal magnetostriction is reduced to half.

THE EFFECT OF THE INTERNAL STRESS ON THE LONGITUDINAL MAGNETOSTRICTION IN WEAK MAGNETIC FIELD (IV)

Shigeo MATSUMAE

(Received March 31, 1959)

〔I〕 Introduction

The present writer studied the effect of the demagnetizing on the longitudinal magnetostriction in weak magnetic field.(1) On further, we study this effect. The experimental results are as follows.

〔II〕 Experimental Results and Discussions

(1) In the case of demagnetizing alternate field of various magnitudes.

The specimens are round bar of nickel, containning a small amount of manganese, 2.5 mm in diameter and 19 cm in length. The specimen are annealed at 900°C for 2 hours and then cooled to the room-temperature with the furnace.

At first, the specimens are demagnetized with the alternate magnetic field having the maximum amplitude of 7 Oersted and then the external compressive stress of 0.2 kg/mm^2 are applied to the specimens. The experimental results are shown in Fig. 1 in the previous paper and Fig. 1-A.

But, when the specimens are again demagnetized with the alternate magnetic field-strength of various magnitudes after the application of load, the longitudinal magnetostriction decreases with the increase of the magnitude of the demagnetizing alternate magnetic field, as shown in Fig. 1 and Fig. 1-B.

In the figures, 0.3, 1, 3 and 7 Oersted are the magnitudes of the maximum amplitude of the demagnetizing alternate magnetic field.

The above tendency that the longitudinal magnetostriction in weak magnetic field is greatly affected by demagnetizing is already described in previous paper.(1)

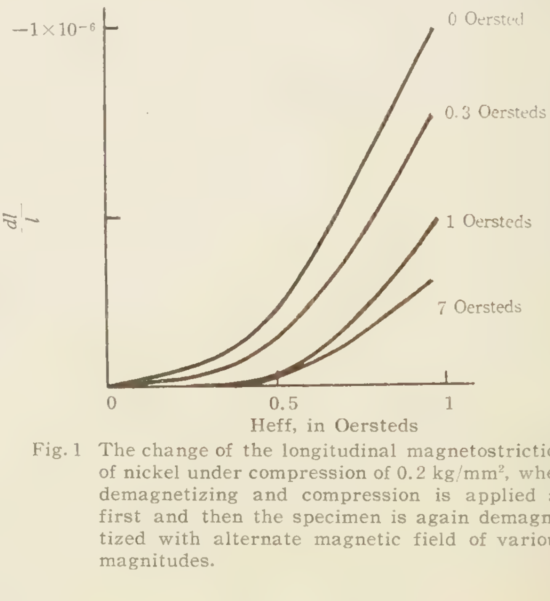


Fig. 1 The change of the longitudinal magnetostriction of nickel under compression of 0.2 kg/mm^2, when demagnetizing and compression is applied at first and then the specimen is again demagnetized with alternate magnetic field of various magnitudes.

This fact indicates that the unstable magnetic domains oriented at about right angle to the demagnetizing field will diminish by demagnetizing.

Especially, in the weak magnetic field the longitudinal magnetostriction is greatly affected by demagnetizing. When the specimen is again demagnetized with the alternate magnetic field of 0.3 Oersted, the longitudinal magnetostriction in weak magnetic field decreases greatly.

From this fact, the longitudinal magnetostriction in weak magnetic field seems to be caused by the unstable domain or domain boundary.

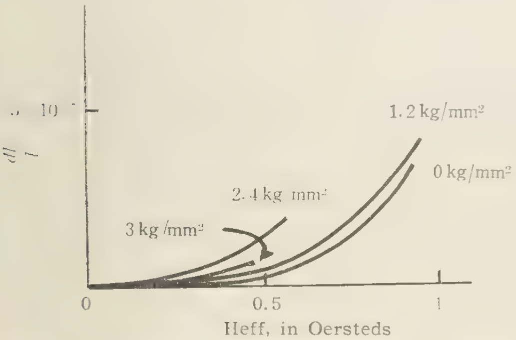


Fig. 2 The longitudinal magnetostriction of Fe Ni (69.93%) alloy under various degrees of tension when demagnetizing and tension is applied at first and then the specimen is again demagnetized with alternate magnetic field of 6 Oersted after application of tension.

(2) In the case of a constant demagnetizing alternate field.

The specimens are round bars of Fe–Ni (69.93%) alloy, 1.5 mm in diameter and 21 cm in length. The specimens of Fe–Co (68.85%) alloy are 2 mm in diameter and 21 cm in length. The specimens are annealed at 900°C for 2 hours and then cooled to the room-temperature with the furnace.

The specimens are at first demagnetized with the alternate magnetic field of 7 Oersted and then the various degrees of the external tensive stress are individually applied to the specimens. The experimental results are shown in Fig. 2–A, and Fig. 7–A, as described in the previous paper.

When the specimens are again demagnetized with the alternate magnetic field of 7 Oersted after the application of the load, the longitudinal magnetostriction decreases in all cases, as shown in Fig. 2–B, Fig. 7–B, Fig. 2 and Fig. 3.

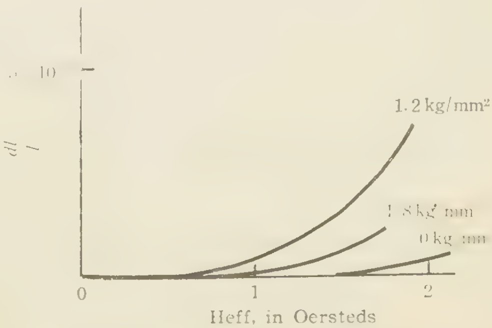


Fig. 3 The longitudinal magnetostriction of Fe-Co (68.85%) alloy under various degrees of tension when demagnetizing and tension is applied at first and then the specimen is again demagnetized with alternate magnetic field of 6 Oersted after application of tension.

In the weak magnetic field the longitudinal magnetostriction increases at first with the increase of the external tensive stress and attains a maximum value and then decreases with the increase of the external tensive stress. This tendency is clear in the figures. This is the same tendency that we observe when

the specimen is not demagnetized after the application of load.

In the weak magnetic field, the boundary displacement between the 90°-domains occurs. Fig. 4 is a highly schematic representation of the magnetic boundaries. When a weak magnetic field in the direction of the arrow is applied, the boundary displacements occur: in the case of (A), the magnetic demain oriented parallel to the direction of the magnetic field is enlarged by the displacement of the domain boundary at the expense of the magnetic domain oriented at the right angle and, on the other hand, in the case of (B), the reverse is the case. Consequently, the longitudinal magnetostriction does not take place in the weak magnetic field.

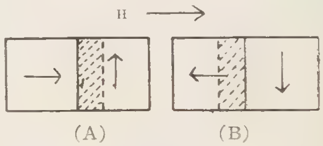


Fig. 4 Domain-interpretation of the effect of a small internal stress on the longitudinal magnetostriction in weak magnetic field.

But, when a weak magnetic field is applied to the specimen which is subjected to a small external compressive or tensive stress in the direction of the magnetizing field according to the positive or negative magnetostriction of the specimen, such a wall-displacement as indicated in (A) is easier than that in (B). Consequently, the longitudinal magnetostriction will take place in the weak magnetic field under the external stress.

On the other hand, when the external tensive or compressive stress applied to the specimen becomes to be large, the unstable magnetic domains oriented at about right angle to the external stress will diminish, as in the case of demagnetizing.

Consequently, at the suitable degree of the external stress, the longitudinal magnetostriction in weak magnetic field will attain a maximum value and then decrease with the increase of the external stress.

(3) Special-case

The specimens are round bars of pure iron, 1.5 mm in diameter and 21 cm in length. The specimens are annealed at 900°C for 2 hours and then cooled to the room-temperature with the furnace. The experimental results are shown in Fig. 5–A and Fig. 5.

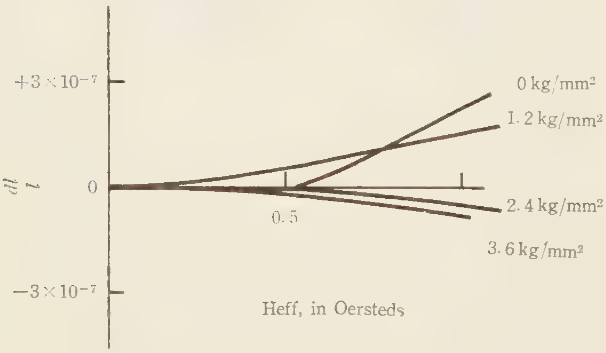


Fig. 5 The longitudinal magnetostriction of Fe under various degrees of tension when demagnetizing and tension is applied at first and then the specimen is again demagnetized with alternate magnetic field of 6 Oersted after application of tension.

The longitudinal magnetostriction is positive in the case of demagnetizing before the application of the load.

But, when the specimens are again demagnetized after the application of the load, the longitudinal magnetostriction under the

external stress of 2.4 kg/mm^2, 3.6 kg/mm^2 and 4.8 kg/mm^2 is negative, as shown in the Fig. 5-B and Fig. 5.

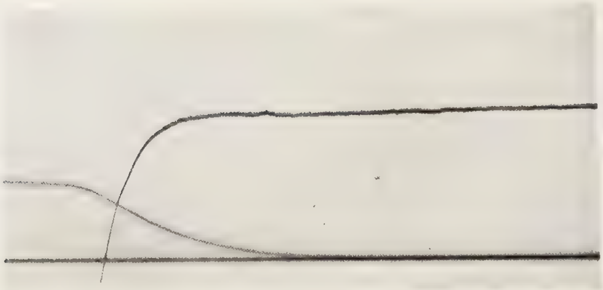
This phenomenon seems to be caused by the unstable domains in the specimen.

Reference

(1) S. Matsumae: "The Effect of Demagnetization on the Longitudinal Magnetostriction in Weak Magnetic Field." Kumamoto Journal of Science Vol. 2 No. 4 1954.

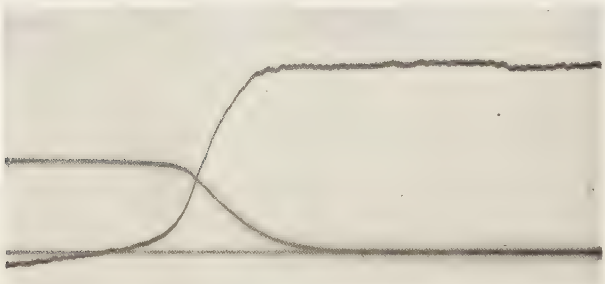
Correction

In the previous paper (The Effect of Demagnetization on the Longitudinal Magnetostriction in Weak Magnetic Field (II)), Fe-Ni(50%) alloy is Fe-Ni(70%) alloy.

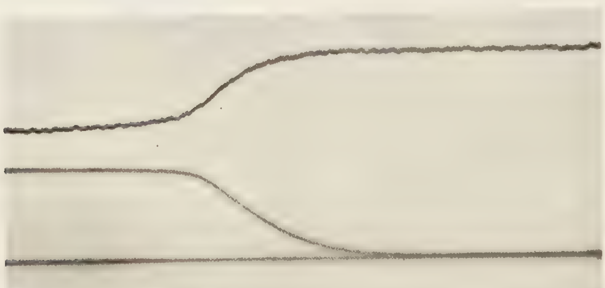


$\sigma = 0$ kg/mm^2

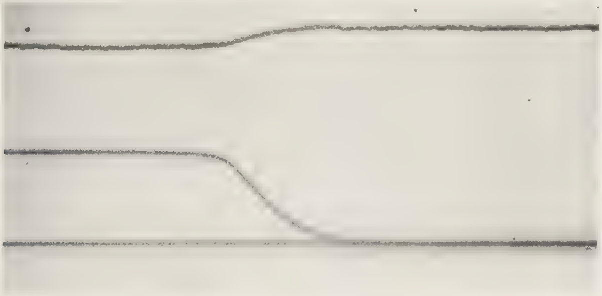
$\sigma = 0.2$ kg/mm^2



$\sigma = 0.4$ kg/mm^2

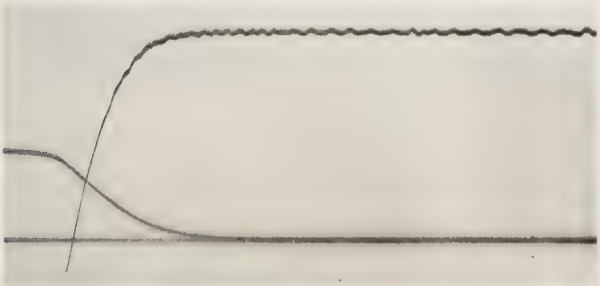


$\sigma = 0.6$ kg/mm^2



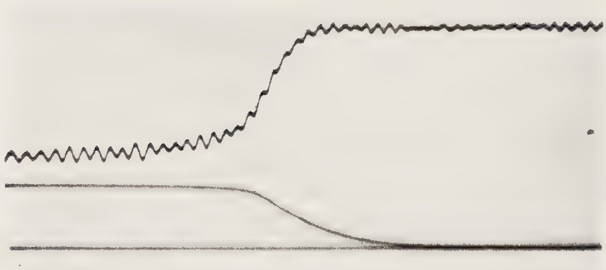
$\sigma = 1.2\mathrm{kg/mm^2}$

Fig. 1-A The oscillograph-records of the longitudinal magnetostriction of Ni under various degrees of compression, when demagnetizing is applied at first and compression subsequently
Sensitivity: 1 cm on the oscillograph-paper corresponds to about 2.5×10^{-7}

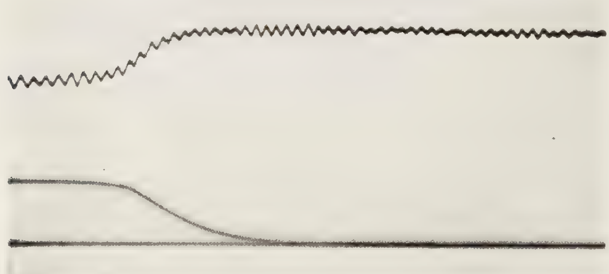


$\sigma = 0.2\,\mathrm{kg/mm^2}$

when the specimen is again demagnetized with the alternate magnetic field having the maximum amplitude of 0.3 Oersted, after demagnetizing and application of compression.



when demagnetized with alternate magnetic field of 1 Oersted

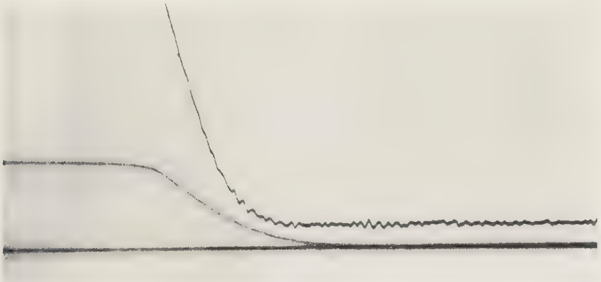


when demagnetized with alternate magnetic field of 7 Oersted

Fig. 1–B The change of the oscillograph-records of the longitudinal magnetostriction of nickel under compression of 0.2 kg/mm^2, when demagnetizing and then compression is applied at first and then the specimen is again demagnetized with alternate magnetic field of various magnitudes.

Sensitivity: 1 cm on the oscillograph-paper corresponds to about 2.5×10^{-7}

$\sigma = 0$ kg/mm^2



$\sigma = 0.6\ \mathrm{kg/mm^2}$

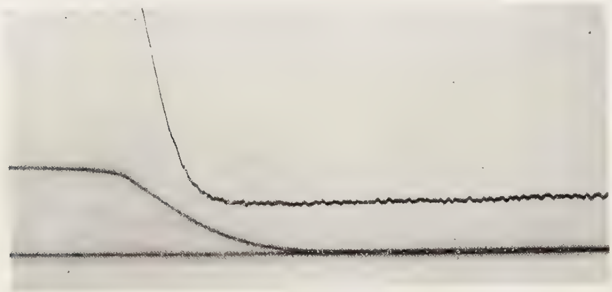
$\sigma = 1.2\ \mathrm{kg/mm^2}$

$\sigma = 2.4\ \mathrm{kg/mm^2}$

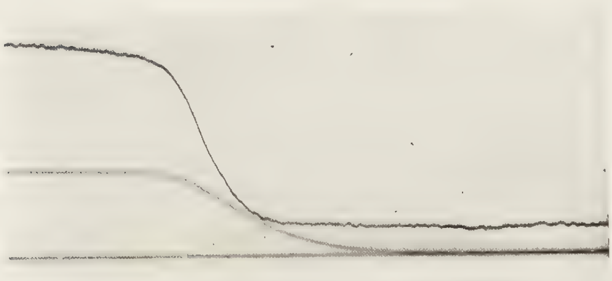
$\sigma = 3.0\ \mathrm{kg/mm^2}$

Fig. 2-A The oscillograph-records of the longitudinal magnetostriction of Fe-Ni (69.93%) alloy under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

Sensitivity: 1 cm on the oscillograph-paper corresponds to about $2 \cdot 10^{-7}$

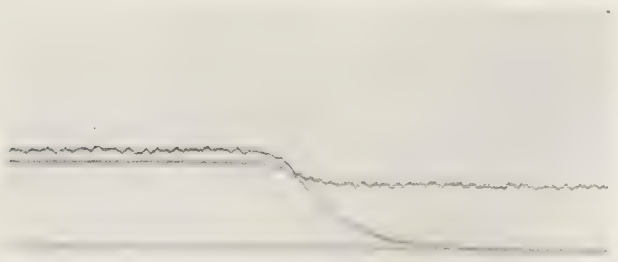


$\sigma = 0\ \mathrm{kg/mm^2}$



$\sigma = 1.2\ \mathrm{kg/mm^2}$

$\sigma = 2.4\ \mathrm{kg/mm^2}$



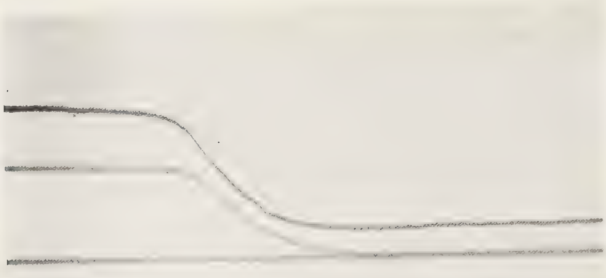
$\sigma = 3.0$ kg/mm^2

Fig. 2-B The change of the oscillograph-records of the longitudinal magnetostriction of Fe-Ni (69.93%) alloy under various degrees of tension, when demagnetizing and tension is applied at first and then the specimen is again demagnetized with alternate magnetic field of 6 Oersted, after tension.

Sensitivity: 1 cm on the oscillograph-paper corresponds to about 2×10^{-7}

$\sigma = 0$ kg/mm^2

$\sigma = 0.6$ kg/mm^2



$\sigma = 1.2\ \mathrm{kg/mm^2}$

$\sigma = 1.8\ \mathrm{kg/mm^2}$

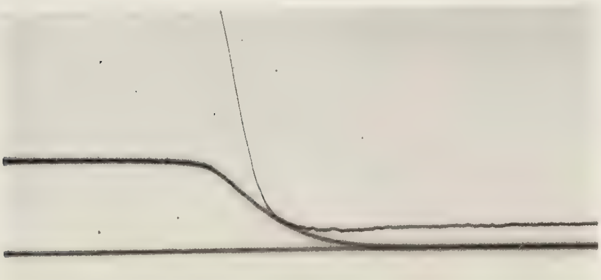
$\sigma = 2.4\ \mathrm{kg/mm^2}$

$\sigma = 3.0\ \mathrm{kg/mm^2}$

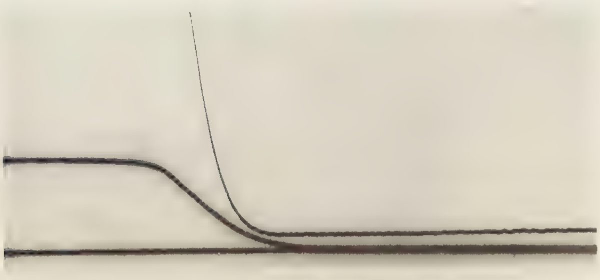
Fig. 3-A The oscillograph-records of the longitudinal magnetostriction of Fe–Ni (50.24%) alloy under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

Sensitivity: 1 cm on the oscillograph-paper corresponds to about $2.5 \cdot 10^{-7}$

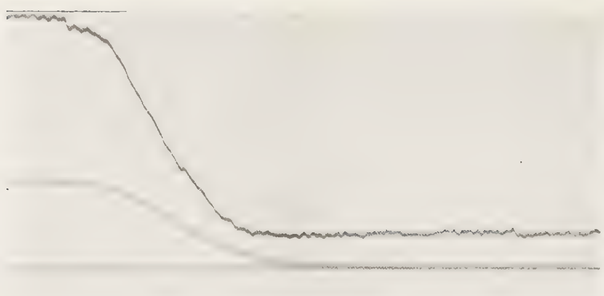
$\sigma = 0\ \mathrm{kg/mm^2}$



$\sigma = 1.2\ \mathrm{kg/mm^2}$



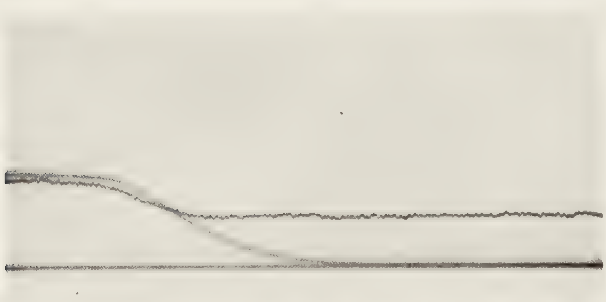
$\sigma = 1.8\ \mathrm{kg/mm^2}$



$\sigma = 2.4$ kg/mm^2

Fig. 4-A The oscillograph-records of the longitudinal magnetostriction of Fe-Ni (20.87%) alloy under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

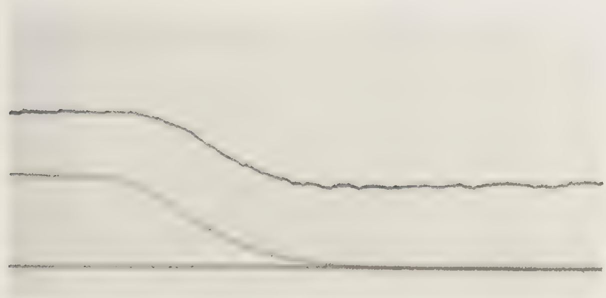
Sensitivity: 1 cm on the oscillograph-paper corresponds to about 2×10^{-7}



$\sigma = 0$ kg/mm^2

$\sigma = 1.2$ kg/mm^2

$\sigma = 2.4$ kg/mm^2



$\sigma = 3.6$ kg/mm^2

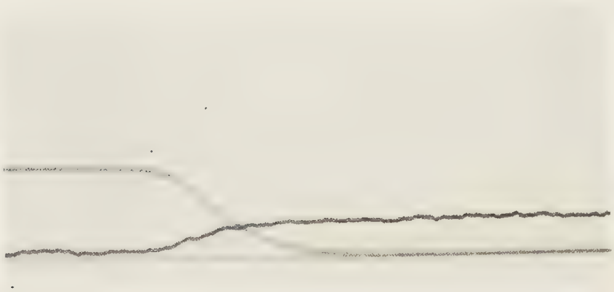
$\sigma = 4.8$ kg/mm^2

Fig. 5-A The oscillograph-records of the longitudinal magnetostriction of iron under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

Sensitivity: 1 cm on the oscillograph-paper corresponds to about 2×10^{-7}

$\sigma = 0$ kg/mm^2

$\sigma = 1.2$ kg/mm^2



$\sigma = 2.4$ kg/mm^2

$\sigma = 3.6$ kg/mm^2

Fig. 5–B The change of the oscillograph-records of the longitudinal magnetostriction of Fe under various degrees of tension, when demagnetizing and tension is applied at first and then the specimen is again demagnetized with alternate magnetic field of 6 Oersted, after tension.

Sensitivity: 1 cm on the oscillograph-paper corresponds to about 2×10^{-7}

$\sigma = 0$ kg/mm^2

$\sigma = 0.6$ kg/mm^2

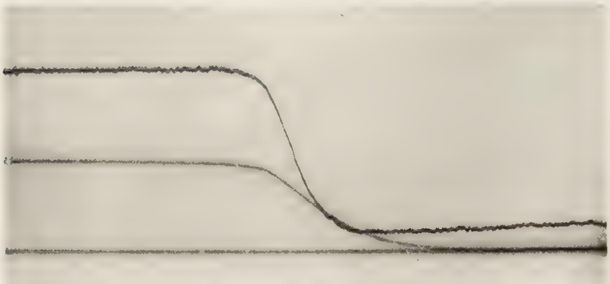
$\sigma = 1.2$ kg/mm

$\sigma = 1.8$ kg/mm^2

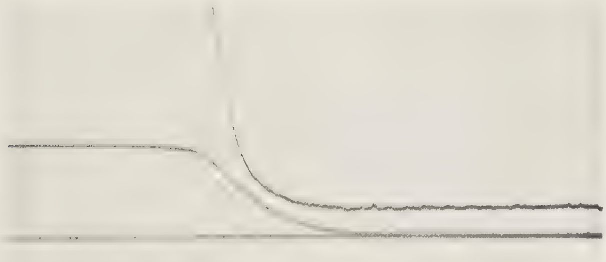
Fig. 6-A The oscillograph-records of the longitudinal magnetostriction of Fe–Co (19.59%) alloy under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

Sensitivity: 1 cm on the oscillograph-paper corresponds to about 2×10^{-7}

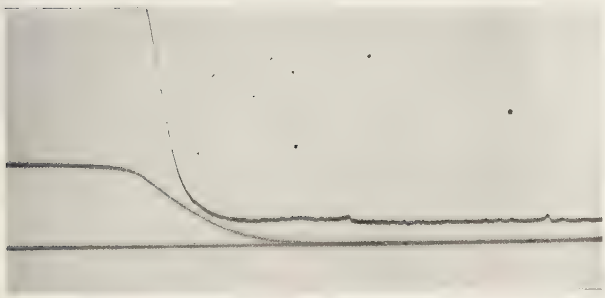
$\sigma = 0$ kg/mm^2



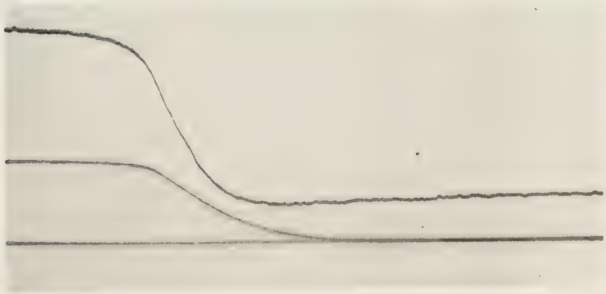
$\sigma = 0.6$ kg/mm^2



$\sigma = 1.2$ kg/mm²



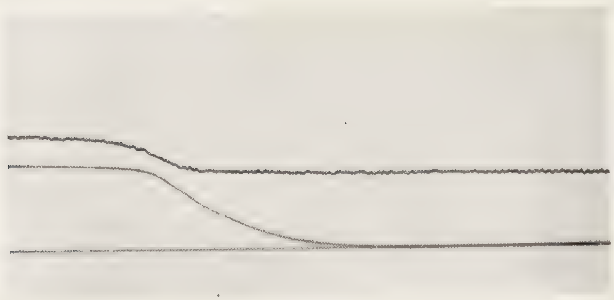
$\sigma = 1.8$ kg/mm²



$\sigma = 2.4$ kg/mm²

Fig. 7-A The oscillograph-records of the longitudinal magnetostriction of Fe-Co (68.85%) alloy under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

Sensitivity: 1 cm on the oscillograph-paper corresponds to about 1.6 10^{-} [illegible]



$\sigma = 0\ \mathrm{kg/mm^2}$

$\sigma = 1.2\ \mathrm{kg/mm^2}$

$\sigma = 1.8\ \mathrm{kg/mm^2}$

Fig. 7–B The change of the oscillograph-records of the longitudinal magnetostriction of Fe–Co (68.85%) alloy under various degrees of tension, when demagnetizing and tension is applied at first and then the specimen is again demagnetized with alternate magnetic field of 6 Oersted after tension.

Sensitivity: 1 cm on the oscillograph-paper corresponds to about $2 \cdot 10^{-}$

$\sigma = 0\ \mathrm{kg/mm^2}$

$\sigma = 0.6\ \mathrm{kg/mm^2}$

Fig. 8-A The oscillograph-records of the longitudinal magnetostriction of Co, when demagnetizing is applied at first and compression subsequently.

Sensitivity: 1 cm on the oscillograph-paper corresponds to about 1.6×10^{-7}

REMARKS ON THE GAUGE TRANSFORMATIONS

Seibun SASAKI

(Received January 30, 1959)

Summary

Gauge transformations in the wide sence are reconsidered from the stand-point of the separation of the transverse and the longitudinal parts of the electromagnetic potentials.

I. As is well known, the equations describing the phenomena concerning the electromagnetic fields are invariant under the gauge transformations. We can consider two kinds of the gauge transformation, that is, the one in the wide sence and the one in the narrow sence[1]. They are defined as follows:

i°) Transformation in the wide sence

(Gauge transformation of the first class)*

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda_1(x) \quad \cdots\cdots (1)$$

where $\Box \Lambda_1(x) = 0$ does not necessarily hold.

ii°) Transformation in the narrow sence

(Gauge transformation of the second class)

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda_2(x) \quad \cdots\cdots (2)$$

where $\Box \Lambda_2(x) = 0$ must hold.

II. Using the electromagnetic four potentials $A_\mu(x)$, Maxwell's fundamental equations can be written as

$$(\Box \delta_{\mu\nu} - \partial_\mu \partial_\nu) A_\nu(x) = -j_\mu(x) \quad \cdots\cdots (3)$$

That the four potentials $A_\mu(x)$ determine the field quantities $F_{\mu\nu}(x)$ by the relations $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the fact that the observable quantities are not A_μ but $F_{\mu\nu}$ allows for A_μ the freedom fo the gauge transformations (1) and (2). Utilizing this degree of freedom, we can employ as the new potentials A'_μ the ones which obey the so-called Lorentz condition

$$\partial_\mu A'_\mu(x) = 0 \quad \cdots\cdots (4)$$

Obviously, the gauge transformation between A_μ and A'_μ is of the first class, because the generating function Λ_1 of this transformation must satisfy the relation

$$\partial_\mu A_\mu + \Box \Lambda_1 = 0 \quad \cdots\cdots (5)$$

*) The nomenclatures in the brackets are of conventional.

Imposing the Lorentz condition for the electromagnetic potentials corresponds to the employment of the special gauge of the first class. Λ_1 itself is ambiguous by the quantity λ which satisfy the relation $\Box\lambda=0$. This ambiguity can not be distinguished from the second class gauge transformation. Therefore, we will thrust the λ-ambiguity into the second class gauge transformation.

III. In the following we will take the special gauge of the second class $\partial_\mu\Lambda_2=0$. Then, the transformation (1) become

$$A_\mu^{0\prime}(x)=A_\mu^0(x)+\partial_\mu\Lambda_1(x) \qquad (6)$$

In the expression (6), A_μ^0 and $A_\mu^{0\prime}$ mean that they do not contain the terms of the form $\partial_\mu\lambda(\Box\lambda=0)$. Using this special gauge of the second class, the relation (5) becomes

$$\Box\Lambda_1+\partial_\mu A_\mu^0(x)=0 \qquad (7)$$

This equation is able to be solved and we get

$$\Lambda_1(x)=-\frac{1}{4\pi}\int G(x,x')\partial'_\mu A_\mu^0(x')d^4x' \qquad (8)$$

where $G(x,x')$ satisfies the relation $\Box G(x,x')=4\pi\delta^4(x-x')$

IV. A_μ^0 can be separated into two parts

$$A_\mu=\mathfrak{A}_\mu+\mathfrak{B}_\mu \qquad (9)$$

where $\mathfrak{A}_\mu$ is the transverse part of the electromagnetic potential and satisfies the relation $\partial_\mu\mathfrak{A}_\mu=0$, $n_\mu\mathfrak{B}_\mu=0$
where n_μ is the time-like unit vector satisfies the relation $n_\mu n_\mu=-1$
$\mathfrak{B}_\mu$ must be constructrd from two parts, the time-like part (scalar part) and the longitudinal part. The former is parallel to the vector n_μ and the latter perpendicular to n_μ and $\mathfrak{A}_\mu$. Therefore, we can assume the form of $\mathfrak{B}_\mu$ as follows[1)]

$$\mathfrak{B}^\mu=n_\mu\partial\Lambda-(\partial_\mu+n_\mu\partial)\Lambda$$

where Λ and Λ' are the scalar functions. Then, we get

$$\Lambda_1(x)=-\frac{1}{4\pi}\int G(x,x')[\partial'^2(\Lambda(x')-\Lambda'(x'))-\Box'\Lambda'(x')]d^4x'$$

or

$$\Lambda_1(x)=\Lambda'(x)-\frac{1}{\Box}\partial^2(\Lambda(x)-\Lambda'(x)) \qquad (10)$$

This expression can be derived directly from the relation $\Box\Lambda_1=-\partial_\mu\mathfrak{B}_\mu(x)$ which can be rewritten as

$$\Box\Lambda_1=\Box\Lambda'(x)-\partial^2(\Lambda(x)-\Lambda'(x)) \qquad (11)$$

V. As is easily seen from the expression (10) or (11), the generating function of the first class gauge transformation can be expressed with the longitudinal and the time-like

parts of the electromagnetic potential. Thus, taking the special gauge in the first class gauge transformation means that some parts of the electromagnetic potential are taken out from the equations containing A_μ. In the other words, the gauge transformation of the first class is unable to be independent of the separation of the longitudinal and transverse parts of the A_μ. These situations are not so important in the non-quantizing theory, because in that theory all the quantities appearing in the equations are c-number and we can treat the generating function of the gauge transformation on the same footing as the field quantities. But the situations are quite different in the quantized theory. In quantizing the theory, two cases are possible;

Case 1. Using the gauge degree of freedom in the c-number theory, one firstly eliminated the longitudinal part of the electromagnetic potential. This procedure corresponds to the so-called "Coulomb Gauge". After taking the Coulomb Gauge, one quantize the transverse part of the electromagnetic potential. This procedure is, at the first glance, nonrelativistic, but could be brought to the relativistic form and, because of there being no longitudinal part which should be quantized and, therefore, making no curious nature occur about the state vector, it may be considered as reasonable procedure.

Case 2. In order to preserve the formal invariance under the Lorentz transformation and the formal equivalence of the each component of the electromagnetic potential, we can adopt the subsidiary condition to the state vector which have, in general, no relation to the c-number theory, i. e. if we would not take a special gauge mentioned in the expression (10) or (11), we could not derive it.

Thus, to assume the Lorentz Condition does not always mean the separation of the electromagnetic potential but we can put them as the same thing by choosing the generating function of the transformation suitably.

VI. As for the second class gauge transformation, there is no necessity of changing the interpretation. The generating function of it is scalar c-number as usual[2]. If we consider the generating function of the second class gauge transformation as an operator, this may introduce a further confusion into the theory. Rather, we might be to consider the second class one as the only gauge transformation and the first class one as the removal of the unwanted part of the electromagnetic potential. Then the case 1 of **V** becomes the special case of the case 2.

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1) S. Ozaki Progr. Theor. Phys. **14** 511 (1955)
S. Hori Soryūshiron Kenkyū (Japanese) 8 334 (1955)
2) For example S. S. Schweber, H. A. Bethe and F. de Hoffmann Mesons and Fields Vol. 1

AN EXPERIMENT AND ITS EXPLORATION CONCERNING SEISMIC REFRACTION METHOD

Ryuzo ADACHI

(Received January 30, 1959)

In the previous Journal, the writer reported a new formula for the exploration concerning seismic refraction method. In this paper we report the result of an experiment concerning seismic refraction method and its exploration by using above formula.

1. Experiment

Place ········· neighbourhood of Misumi town Kumamoto prefecture

Date ········· Nov. 22, 1958

Fig. 1 shows the section of the ground where the experiment was done and the transducers were settled at p_1, p_2, ··············· and $0, 0'$ denote the explosion points. Fig. 2 is the record of this experiment by oscillograph.

2. Time-Distance Curve

At first, from the record of oscillograph, we draw the time-distance curve (Fig. 3) which shows the relation between the horizontal distances of pick-ups from the explosion points and the travelling-times.

Next, from this curve, we draw the time-distance curve $t=\varphi(X)$ and $\bar{t}=\bar{\varphi}(\bar{X})$ (Fig. 4) which shows the relation of time-distance when the pick-ups and the explosion points were settled on the straight line $00''$, where $0''$ is the imaginary explosion point corresponding to $0'$.

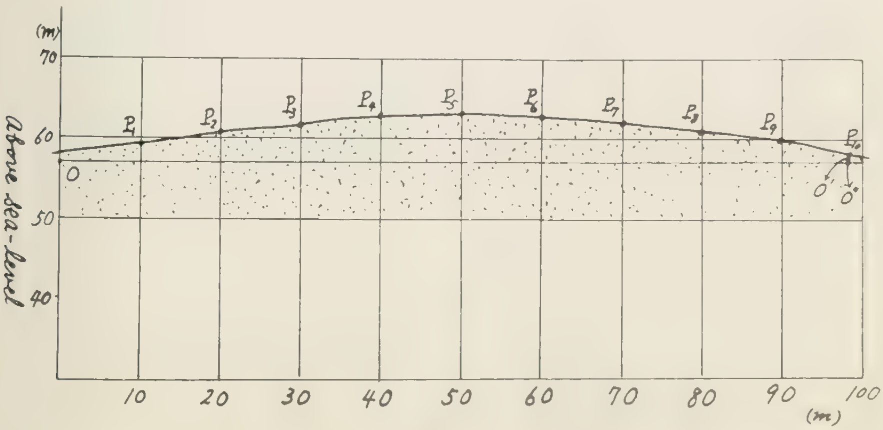


Fig. 1

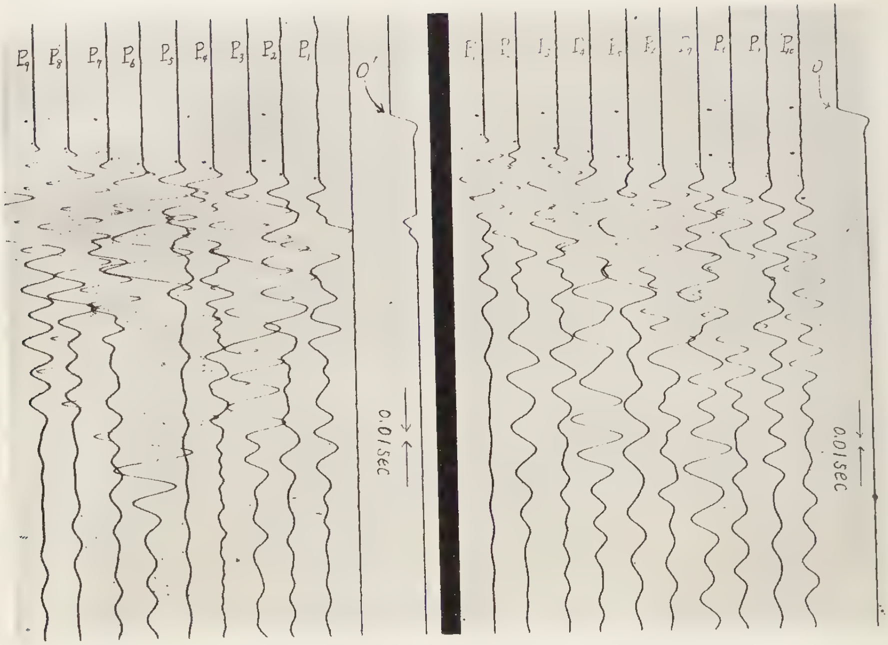


Fig. 2

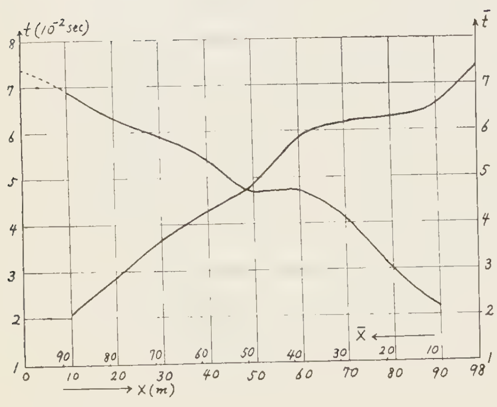
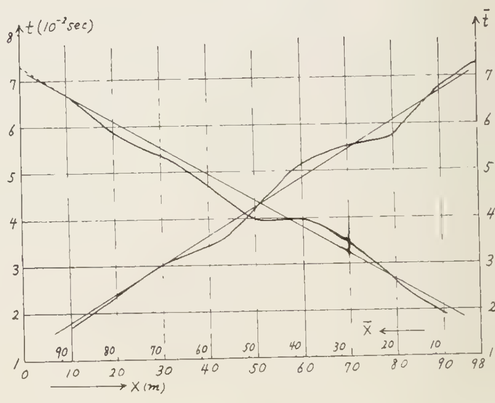


Fig. 3



F.g . 4

3. Determination of v_2 and $y(x)$

Straight lines near the cueves $t=\varphi(X)$, $\bar{t}=\bar{\varphi}(\bar{X})$ are

$$\left.\begin{aligned} t&=\varphi_0(X)=0.00062X+0.012 \\ \bar{t}&=\bar{\varphi}_0(\bar{X})=0.00057\bar{X}+0.015 \end{aligned}\right\} \qquad \text{(i)}$$

and solving them by ordinary method, we get

$$v_2 \doteqdot 1.7\text{km/sec}, \quad y(x) \doteqdot 0.0087x + 4.3$$

Where $v_1 = 0.35\text{km/sec}$, and we take the formulas when $\beta = 0$.

At first we have

$$\alpha_0 = 0.35/\sqrt{(1.7)^2 - (0.35)^2} \doteqdot 0.213$$

$$C_0 = y(30) \doteqdot 4.6^{(m)} \qquad \therefore \quad \alpha_0 C_0 \doteqdot 1^{(m)}$$

where we put $m = 30^{\text{m}}$.

In this case we obtain

$$\gamma = k = \bar{k} = 1,\ A = \bar{A} = \alpha C,\ l = 98^{\text{m}},\ B = 30 + \alpha C,\ \bar{B} = 68 + \alpha C$$

and putting $n = 85^{\text{m}}$ gives

$$\left.\begin{aligned} \alpha &= 3.182k\ \{\varphi(85+\alpha C) + \bar{\varphi}(68+\alpha C) - \varphi(30+\alpha C) - \bar{\varphi}(13+\alpha C)\} \\ C &= \frac{175}{k}\ \{\varphi(30+\alpha C) + \bar{\varphi}(68+\alpha C) - \bar{\varphi}(98)\} \end{aligned}\right\} \cdots \quad \text{(ii)}$$

and solving these equations by iterative method we get

n	0	1	2	3	4
α_n	0.213	0.179	0.207	0.207	0.207
$C_n^{(m)}$	4.6	2.05	1.95	1.94	1.94

and $\alpha = 0.207$, $C = 1.94^{\text{m}}$

therefore $v_2 = 1.73\text{km/sec}$, $\alpha C = 0.4^{\text{m}}$, $v_1 k = 357\text{m/sec}$.

Accordingly we get following formulas:

$$\left.\begin{aligned} &\text{when } x + 0.4 \geqq a \doteqdot 10 \\ &\qquad y(x) = 357\varphi(x+0.4) - 0.207x - 2.75 \\ &\text{when } x + 0.4 < a \\ &\qquad y(x) = 357\bar{\varphi}(98.4 - x) + 0.207x - 23.50 \end{aligned}\right\} \cdots\cdots\cdots\cdots \quad \text{(iii)}$$

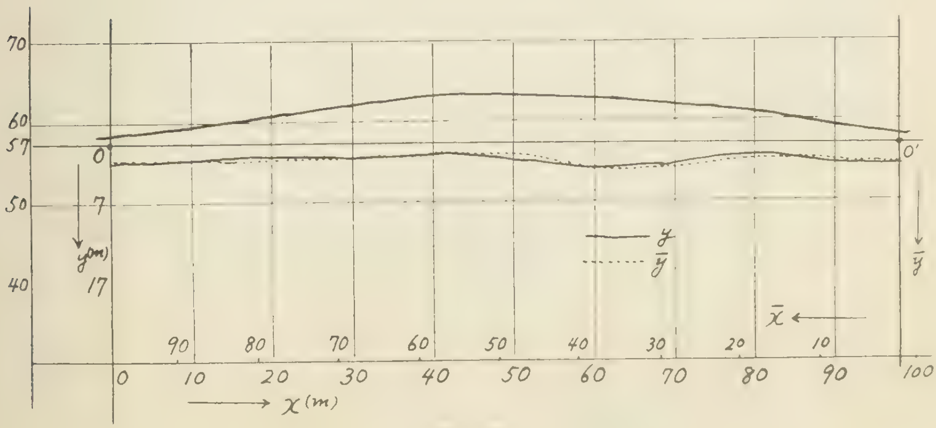


Fig. [illegible]

Similarly

when $\bar{x}+0.4 \geqq \bar{a} \doteqdot 8$

$$\left.\begin{array}{l} \bar{y}(\bar{x})=357\bar{\varphi}(\bar{x}+0.4)-0.207\bar{x}-3.20 \\ \text{when} \quad \bar{x}+0.4<\bar{a} \\ \bar{y}(\bar{x})=357\varphi(98.4-\bar{x})+0.207\bar{x}-23.04 \end{array}\right\} \cdots\cdots\cdots\cdots \text{(iv)}.$$

Applying the values of $\varphi(X)$, $\bar{\varphi}(\bar{X})$ obtained from Fig. 4 to (iii) and (iv), we get the graph shown in Fig. 5, and it shows that $y(x) \doteqdot \bar{y}(l-x)$.

Reference

(1) R. Adachi: A method of exploration on the seismic prospecting. Kumamoto J. S. Vol. 3, No. 1, 1957.

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熊 本 大 学 理 学 部

ON SIMILARITIES AND TRANSITIVE ABELIAN GROUPS OF MOTIONS IN FINSLER SPACES

Yasuo NASU

(Received June 1, 1959)

Introduction. In the paper we deal with a complete Finsler space of class C^1 which admits a non-isometric similarity or a transitive abelian group of motions. We showed in [1][1] that, if a complete Finsler space of class C^1 admits a non-isometric and global similarity, the space is Minkowskian or isometric to Minkowskian according as its indicatrices, which are unit spheres of tangent Minkowskian spaces, are convex or not necessarily convex. But we did not deal with the case where a Finsler space admits a non-isometric and local similarity. Hence we discuss first the properties of such a Finsler space and second the properties of a Finsler space which admits a transitive abelian group of motions.

H. Busemann showed in [3] that, if a metric space called a G-space admits a transitive abelian group of motions, the space is locally Minkowskian and homeomorphic to the topological product of the finite number of real lines and circles. Under a suitable assumption of differentiability the above also holds for a complete Finsler space. But, if the space is of class C^1, the above is not obvious. Hence we show the above under a reasonable assumption.

§ 1. Let R be an n-dimensional complete Finsler space of class C^1 with integrand $F(x, \dot{x})$ $(n \geqq 2)$. In a coordinate neighborhood U a point with coordinates $(x^1, \cdots\cdots, x^n)$ will be denoted by x and a contravariant vector with components $(\xi^1, \cdots\cdots, \xi^n)$ by ξ. Suppose that the function $F(x, \xi)$ is continuous in the variables x^i's and ξ^i's and satisfies the following conditions:

i) $F(x, \xi) > 0$ *for any vector* $\xi \neq 0$.

ii) $F(x, c\xi) = cF(x, \xi)$ *for any positive number* c.

Suppose further that for any two points there exists an arc which connects these points. Then, by integral method, the length $l_F(C)$ of a curve C of class D' from a point p to a point q is defined, and the distance $\rho(p, q)$ is defined by the greatest lower bound of the lengths of those curves from p to q. By use of the distance ρ we can again define the length $l_\rho(E)$ of a continuous curve E. Obviously $l_\rho(C) \leqq l_F(C)$. The equal sign always holds when and only when the indicatrices are convex.

Let Ψ be a transformation of R on itself such that, if $p\Psi = p'$ and $q\Psi = q'$,

$$\rho(p', q') = k\rho(p, q),$$

where k is a positive constant not equal to 1. Then Ψ is said a non-isometric similarity or simply a global similarity of the space R on itself. If for a point p a positive

1) Numbers in brackets refer to the references cited at the end of the paper.

number δ_p exists such that

$$\rho(x\Psi, y\Psi) = k\rho(x, y) \ (k > 0, \neq 1)$$

holds for any two points x and y in the sphere neighborhood $S(p, \delta_p)$ $(=\{x \mid \rho(p, x) < \delta_p\},\ \delta_p > 0)$, Ψ is said a local similarity of the space R on itself.

If the space R admits a global similarity Ψ on itself then the space is isometric to a Minkowskian space. If further the indicatrices: $F(x, \xi) = 1$ are convex, the space is Minkowskian [1].

§ 2. Let $\tilde{R}$ be the universal covering space of R and Ω a covering transformation of $\tilde{R}$ onto R. Then we have

(2.1) *If the space R admits a local similarity Ψ on itself, then the space $\tilde{R}$ admits a global similarity $\tilde{\Psi}$ on itself such that $\tilde{\Psi}\Omega = \Omega\Psi$.*

Proof. Let p be a point of R and put $p\Psi = p'$. Suppose further that the universal covering space $\tilde{R}$ was constructed by choosing the point p as origin. Hence $\tilde{p} = p$ and we can put $\tilde{p}\Omega = p$.

Let x be a point of R and put $x\Psi = x'$, and further let C be an arc from p to x. Then $C\Psi$ is a continuous curve from p' to x'. We denote this curve by C'. Let D be an arc from p to p', $\tilde{D}$ the arc which lies over D and issues from $\tilde{p}$ and $\tilde{C}'$ the continuous curve which issues from the end point $\tilde{p}'$ of $\tilde{D}$ and lies over C'. Then the end-point $\tilde{x}'$ of $\tilde{C}'$ lies over x'. Let $\tilde{C}$ be an arc which lies over C and issues from $\tilde{p}$ and $\tilde{x}$ its end-point. The point $\tilde{x}$ lies over x.

By virtue of the assumption the sphere neighborhood $S(x, \delta_x)$ is mapped onto $S(x', \delta'_x)$ $(\delta'_x = k\delta_x)$ under the local similarity Ψ. On the other hand there exists a positive number ρ_x not greater than δ_x such that $S(\tilde{x}, \rho_x)$ and $S(\tilde{x}', \rho'_x)$ $(\rho'_x = k\rho_x)$ are under Ω isometrically mapped onto $S(x, \rho_x)$ and $S(x', \rho'_x)$ respectively. From this we see that there exists a similarity $\tilde{\Psi}$ of $S(\tilde{x}, \rho_x)$ onto $S(\tilde{x}', \rho'_x)$ such that

$$\rho(\tilde{y}\tilde{\Psi}, \tilde{z}\tilde{\Psi}) = k\rho(\tilde{y}, \tilde{z}) \text{ for two points } \tilde{y} \text{ and } \tilde{z} \text{ in } S(\tilde{x}, \rho_x)$$

and further

$$\rho(\tilde{y}\tilde{\Psi}\Omega, \tilde{z}\tilde{\Psi}\Omega) = \rho(\tilde{y}\Omega\Psi, z\Omega\Psi) = k\rho(y, z),$$

where $\tilde{y}\Omega = y$ and $\tilde{z}\Omega = z$. It is easy to see that the similarity $\tilde{\Psi}$ can be extended to the whole space $\tilde{R}$. Obviously $\tilde{\Psi}$ is a local similarity of $\tilde{R}$ on itself. Hence the proposition is proved by showing that $\tilde{\Psi}$ is global. To do this we prove the following

(2.2) *If the Finsler space R is simply connected, then the local similarity Ψ is global.*

Proof. Let G be a shortest connection from a point p to a point q and put

$$p' = p\Psi, \ q' = q\Psi \text{ and } G\Psi = G'.$$

Then G' is a continuous curve from p' to q' and we see from the definition of local similarity

$$kl_\rho(G)=k\rho(p, q)=l_\rho(G').$$

If $l_\rho(G')=\rho(p', q')$, the proposition is clear. Hence we assume that there exists a shortest connection G'' from p' to q' such that

$$\rho(p', q')=l_\rho(G'')<l_\rho(G').$$

Let $U\times V$ be the topological product of the segments U: $0\leqq u\leqq 1$ and V: $0\leqq v\leqq 1$. Since R is simply connected, there exists a continuous mapping of $U\times V$ into R such that $f(0,0)=p'$, $f(1,1)=q'$ and $f(u, 0)$, $0\leqq u\leqq 1$, and $f(u, 1)$, $0\leqq u\leqq 1$, coincide with G' and G'' respectively.

Let $g_0(u)$, $0\leqq u\leqq 1$, be the parametrization of G such that

$$g_0(u)\Psi=f(u, 0) \text{ for } 0\leqq u\leqq 1.$$

To simplify the notation we put $\rho_{g_0(u)}=\rho_0(u)$. Then there exists a subdivision of the interval $[0,1]$: $0=u_0<u_1<\cdots\cdots<u_m=1$ such that each $S(g_0(u_i), \rho_0(u_i))$ is mapped onto $S(f(u_i, 0), \rho'_0(u_i))$ $(\rho_0(u_i)k=\rho'_0(u_i))$ under Ψ and

$$S(g_0(u_i), \rho_0(u_i))\cap S(g_0(u_{i+1}), \rho'_0(u_{i+1}))\neq\phi$$
$$(i=0, 1,\cdots\cdots, m-1).$$

From this there exists a positive number v_1 such that

$$f(u, v_1)\subset S(f(u_i, 0), \rho'_0(u_i))$$

for $u_i\leqq u\leqq u_{i+1}$ $(i=0, 1,\cdots\cdots, m-1)$ and for $u_{i-1}\leqq u\leqq u_i (i=1,\cdots\cdots, m)$. By considering the contraction of Ψ to each $S(g_0(u_i), \rho_0(u_i))$ we have a continuous curve $g_1(u)$, $0\leqq u\leqq 1$, from p to q such that

$$g_1(u)\Psi=f(u, v_1),\ 0\leqq u\leqq 1, \text{ and}$$
$$g_1(u)\subset S(g_0(u_i), \rho_0(u_i))$$

for $u_i\leqq u\leqq u_{i+1} (i=0, 1,\cdots\cdots, m-1)$ and for $u_{i-1}\leqq u\leqq u_i (i=1,\cdots\cdots, m)$. Under the same consideration we have from $g_1(u)$, $0\leqq u\leqq 1$, and $f(u, v_1)$, $0\leqq u\leqq 1$, a continuous curve $g_2(u)$, $0\leqq u\leqq 1$, such that

$$g_2(u)\Psi=f(u, v_2) \text{ for } 0\leqq u\leqq 1,$$

where $0<v_1<v_2\leqq 1$. By repeating this process we have after finite steps a continuous curve $g_k(u)$, $0\leqq u\leqq 1$, from p to q such that

$$g_k(u)\Psi=f(u, 1) \text{ for } 0\leqq u\leqq 1.$$

We denote by H this continuous curve. It is easy to see from the above that the length of the curve H equals $l_\rho(G'')/k$. Thus we have

$$kl_\rho(H)=l_\rho(G'')<l_\rho(G')=kl_\rho(G).$$

We therefore have $l_\rho(H)<l_\rho(G)$. But this is a contradiction. From this it follows that Ψ is a global similarity.

If the space R admits a local similarity Ψ on itself, the universal covering space $\tilde{R}$ admits a global similarity $\tilde{\Psi}$ such that $\tilde{\Psi}\Omega=\Omega\tilde{\Psi}$. The space $\tilde{R}$ is Minkowskian [1].

If R is compact, then R does not admit a local similarity with dilation factor less than 1. If the dilation factor k is less than 1, the space is simply connected. Hence Ψ is global. Since there exists on $\tilde{R}$ only one fixed point under $\tilde{\Psi}$ there exists also on R only one fixed point. Thus we have from [1] the following

(2.3) **Theorem.** *Let R be a complete Finsler space of class C^1 with convex indicatrices. If R admits a local similarity Ψ, the space is Minkowskian or locally Minkowskian according as its dilation factor k is less than 1 or greater than 1. Then the space R has only one fixed point p under Ψ and is isometric or locally isometric to the tangent Minkowskian space at the point p according as the above two cases.*

The arguments apply to a Riemannian space which admits a local similarity, i. e., we have the following

(2.4) **Theorem.** *Let R be a complete Riemannian space of class C^1. If R admits a local similarity, then the space is Euclidean or locally Euclidean according as its dilation factor is less than 1 or greater than 1.*

At the end we consider the case where the indicatrices of the space R are not necessarily convex. Let us denote by γ_x the indicatrix: $F(x, \xi)=1$ at a point x. Let $\bar{\gamma}_x$ be the convex closure of γ_x. Then $\bar{\gamma}_x$ is represented as $\bar{F}(x, \xi)=1$ by choosing a continuous function in the variables $x^{i'}s$ and $\xi^{i'}s$ such that $\bar{F}(x, \xi)$ is positive for $\xi\neq0$ and positively homogeneous of first order in the variables $\xi^{i'}s$. Let $\bar{R}$ be the Finsler space with the indicatrices $\bar{\gamma}_x$ instead of γ_x. Then the space $\bar{R}$ is isometric to R [2]. Hence, if R admits a local similarity Ψ, the space $\bar{R}$ also admits the local similarity Ψ. From this we have the following

(2.5) **Theorem.** *Let R be a complete Finsler space of class C^1 with not necessarily convex indicatrices. If R admits a local similarity Ψ on itself, the space is Minkowskian or locally Minkowskian according as its dilation factor is less than 1 or greater than 1. Then the space has only one fixed point p under Ψ and is isometric or locally isometric to the tangent Minkowskian space T_p at the point p according as the above two cases.*

§3. In this paragraph we deal with a Finsler space R of class C^1 which admits a transitive abelian group of motions Γ. For two points x and y there exists a motion $\Psi(\in\Gamma)$ such that $y=x\Psi$. If $\Phi\in\Gamma$, we have then

$$\rho(x, x\Phi)=\rho(x\Psi, x\Phi\Psi)=\rho(x\Psi, x\Psi\Phi)=\rho(y, y\Phi).$$

It follows from this that every element of Γ has no fixed point unless it is identity E. Hence Γ is simply transitive.

We topologize the group Γ by defining the distance of two elements Ψ and Φ such that $\sigma(\Psi, \Phi)=\rho(x\Psi, x\Phi)\,(x\in R)$. Let $\{\Phi_n\}$ be a sequence of elements of Γ. If for a positive number ε a positive integer N exists such that $\sigma(\Phi_m, \Phi_n)<\varepsilon$ for $m, n\geq N$, there exists a motion Φ of Γ such that $lim_{n\to\infty}\sigma(\Phi_n, \Phi)=0$. Hence the group Γ is closed. We first prove the following

(3.1) **Theorem.** *Let R be a complete Finsler space of class C^1 with not necessarily*

convex indicatrices. If R admits a transitive abelian group of motions Γ and $(x, x\Phi, x\Phi^2)$[2] *for a point x and for a motion $\Phi(\neq E)$ of Γ, then the space is isometric to Minkowskian.*

Proof. Let p be a point of R and U a coordinate neighborhood of p. Let δ be a positive number such that $\overline{S(p, \delta)} \subset U$. Further let $\mathfrak{l}$ be a Euclidean half straight line issuing from p and x the first point on $\mathfrak{l}$ such that $\rho(p, x)=\delta$. The totality of such points x coincides with the boundary $K(p, \delta)$ of $S(p, \delta)$. Thus we see that $K(p, \delta)$ is homeomorphic to an $(n-1)$-dimensional Euclidean sphere.

As we said at the begining of this paragraph, the group Γ is simply transitive. Hence for a point x of $K(p, \delta/2)$ there exists a motion Φ of Γ such that $p\Phi=x$. The point $x\Phi$ clearly lies on $K(p, \delta)$, since $(p, p\Phi, p\Phi^2)$. The totality $\{x\Phi\}$ of such points $x\Phi$ is homeomorphic to $K(p, \delta/2)$ and therefore coincides with $K(p, \delta)$, We put $x\Phi=x_0, x=x_1$. Under the same consideration there exists a point x_2 of $K(p, \delta/2^2)$ and a motion Ψ of Γ such that

$$p\Psi=x_2,\ p\Psi^2=x_2\Psi=x_1 \text{ and } \Psi^2=\Phi.$$

If we put $x_1\Psi=x_3$, then we have $(p, x_2, x_1), (x_2, x_1, x_3), (x_1, x_3, x_0)$ and (p, x_1, x_0). We denote by L_1 the Euclidean polygon $\{p, x_1, x_0\}$ and by L_2 the Euclidean polygon $\{p, x_2, x_1, x_3, x_0\}$. Repeating this process, we have a sequence of Euclidean polygons $\{L_n\}$. The sequence $\{L_n\}$ clearly converges to a shortest connection G from p to x_0 which is called a geodesic arc [3].

Obvoiusly the geodesic arc G is reversible and its prolongation is locally possible and unique under motions of Γ. We denote by T the whole prolongation. T is called a geodesic and denote by $\mathfrak{F}_p$ the totality of such geodesics. Two geodesics of $\mathfrak{F}_p$ have no common points except the point p. Next we prove this.

Suppose that there exist two geodesics T_1 and T_2 of $\mathfrak{F}_p$ which intersect at a point q different from p. Let G_1 and G_2 be the subarcs of T_1 and T_2 from p to q respectively and suppose further that q is the first common point of T_1 and T_2 from p. Then $l_\rho(G_1)=l_\rho(G_2)$ and there exist a positive integer $m(\geq 2)$ and the two points p_1, p_2 on G_1, G_2 such that $l_\rho(G_1)/2^m=s$ and $\rho(p, p_1)=\rho(p, p_2)=s$. By the assumption the points p_1 and p_2 do not coincide. If we put $p\Psi=p_1$ and $p\Phi=p_2$, then it is easy to see

$$p\Psi^{2m}=p\Phi^{2m}=q, (p, p\Psi^{2m-1}, p\Psi^{2m}) \text{ and } (p, p\Phi^{2m-1}, p\Phi^{2m}).$$

We have from this

$$\Psi^{2m}\Phi^{-2m}=E.$$

On the other hand, since $\Psi\Phi^{-1}\neq E$, we also have $\Psi^{2m}\Phi^{-2m}\neq E$, which contradicts the above.

Let $\{a_n\}$ and $\{b_n\}$ be sequences of points which converges to a point a and a point b respectively and for each n G_n the geodesic of $\mathfrak{F}_{a_n}$ through the point b_n. Then the closed limit of the sequence of geodesics $\{G_n\}$ coincides with the geodesic of $\mathfrak{F}_a$ through the point b. The set of all points on the geodesics of $\mathfrak{F}_p$ forms an open and

2) If for three points x, y and z $\rho(x, y)+\rho(y, z)=\rho(x, z)$, then we denote this by (x, y, z).

closed set and therefore coincides with the space R.

From the above we see that the geodesics of $\mathfrak{F}_p$ simply covers the whole space R except p. Let $\mathfrak{F}$ be the totality of such systems $\mathfrak{F}_x(x \in R)$. If $p\Omega=x$, $\Omega \in \Gamma$, then we have $\mathfrak{F}_p\Omega=\mathfrak{F}_x$. To prove the theorem we show the following

(3.2) *Let a, b and c be three points which do not lie on a geodesic of $\mathfrak{F}$ and b' and c' the points on the geodesic arcs which connect a, b and a, c respectively and such that $2\rho(a, b')=\rho(a, b)$ and $2\rho(a, c')=\rho(a, c)$. Then $\rho(b, c)=2\rho(b', c')$.*

If in the proposition $2\rho(a, b')=\rho(a, b)$ and $2\rho(a, c')=\rho(a, c)$, then $2\rho(b, b')=\rho(b, a)$ and $2\rho(c, c')=\rho(c, a)$. This is clear from the above proof. If further the proposition is proved, it is also clear that $2\rho(c', b')=\rho(c, b)$. Next we prove the proposition.

Proof. We put $a\Psi=b'$, $a\Phi=b'$ and $b'\Omega=c'$. Then the following is clear:

$$\Psi\Omega=\Phi,\ a\Psi^2=b \text{ and } a\Phi^2=c.$$

If we further put $d=a\Psi\Phi\Psi$, then

$$b\Omega=a\Psi^2\Omega=a\Psi\Phi\Psi=d \text{ and}$$
$$b\Omega^2=a\Psi^2\Omega^2=a(\Phi\Omega^{-1})^2\Omega^2=a\Phi^2=c.$$

Hence the point d lies on the geodesic arc of $\mathfrak{F}$ connecting the points b and c and is the mid-point of these points. It is easy to see

$$c'\Psi=a\Phi\Psi=a\Psi\Omega\Psi=d \text{ and } b'\Psi=b.$$

Hence $\rho(b', c')=\rho(b'\Psi, c'\Psi)=\rho(b, d)=\rho(b, c)/2$, from which the proposition follows.

Let x be a point of R and Ψ the motion of Γ such that $p\Psi=x$. We put $x\Psi=x'$ and denote by $x'=x\Theta$ this correspondence: $x \to x'$. In such a way we define a transformation Θ of R on itself. Obviously Θ is a non-isometric similarity with dilation factor 2. The space admits the similarity Θ and the point p is fixed under Θ. Hence the space is isometric to Minkowskian.

Remarks. If in a Minkowskian space the spheres are convex, all straight lines are geodesics. Even if the spheres are not convex, all straight lines are geodesics in the sense of l_ρ-length but not in the sense of l_F-length. It is to be noticed that in the theorem the assumption of the convexity of indicatrices is not necessary. If in the theorem the indicatrices are convex, the space is clearly a Minkowskian space.

Let Ψ be a motion of R on itself and a point x be carried into a point y under Ψ. Let U and V be coordinate neighborhoods of x and y and $(x^1, \cdots\cdots, x^n)$ and $(y^1, \cdots\cdots, y^n)$ their coordinates respectively. Then the relation: $y=x\Psi$ can be represented in the following form:

$$y^i=f^i(x^1, \cdots\cdots, x^n)\ (i=1, \cdots\cdots, n).$$

Obviously the functions $f^i(x^1, \cdots\cdots, x^n)$ are continuous in the variables x^i's. If each $f^i(x^1, \cdots\cdots, x^n)$ has continuous derivatives of first order with respect to the variables x^i's. The motion Ψ is said to be of class C^1.

(3.3) **Theorem.** *Let R be a complete Finsler space of class C^1 with convex indicatrices. If R admits a transitive abelian group of motions Γ such that each of Γ is of class C^1, the space is locally Minkowskian and homeomorphic to the topological product of the finite number of real lines and circles.*

If in the above theorem the space is of 2-dimensions, then the space is either a plane, a cylinder or a torus with Minkowskian metric. Next we prove the theorem.

Proof. As we said at the begining of this paragraph, the group Γ is simply transitive. Let p be a point of R, $\mathfrak{l}$ the half straight line issuing from p and $\lambda\,(=(\lambda^1, \cdots\cdots, \lambda^n))$ is the Euclidean unit vector with the direction of $\mathfrak{l}$. If a point q on $\mathfrak{l}$ tends to p, we have by virtue of the convexity of indicatrices

$$lim_{q\to p}\frac{\rho(p, q)}{F(p, \lambda)\, e(p, q)}=1.$$

Let x be a point of R and $p\Psi=x$, $\Psi\in\Gamma$. Further let U_x be a coordinate neighborhood of x and y a point of U_x such that the Euclidean segment $E(x, y)$ is contained in U_x. Then the motion Ψ carries the Euclidean segment $E(p, q)$, whose length equals 1 and direction is λ, into an arc L of class C^1. Let $\{r_n\}$ be a sequence of points of L which converges to the point x and λ_n the Euclidean unit vector of each $E(x, r_n)$. Then the sequence of Euclidean unit vectors $\{\lambda_n\}$ converges to a vector μ_x, and we have by putting $q_n=r_n\Psi^{-1}(n=1, 2, \cdots\cdots)$

$$lim_{n\to\infty}\frac{F'(x, \lambda_n)\, e(x, r_n)}{F(p, \lambda)\, e(p, q_n)}=1, \tag{3.4}$$

where $e(x, y)$ is the Euclidean distance between two points x and y and $F'(x, \dot{x})$ denotes the integrand in the coordinate neighborhood U_x. From this it follows that the correspondence: $\lambda\to\mu_x$ is one-to-one and bicontinuous and the tangent Minkowskian spaces T_p and T_x are isometric in the above directions λ and μ_x.

Let z be a point of $E(x, y)$ and μ_z the Eulcidean unit vector at z which corresponds to λ as in the above. We show that μ_x and μ_y are parallel.

Let $E(x, x')$ be the Euclidean segment with length 1. There exist motions Φ and Σ of Γ such that $x'=x\Phi, y=x\Sigma$; $\Phi, \Sigma\in\Gamma$. Then the images of $E(x, x')$ under Σ and of $E(x, y)$ under Φ are arcs with the common end-point, since the group Γ is abelian. Let x'' be the point of $E(x, x')$ and Φ' the motion of Γ such that $x''=x\Phi'$. Then the images of $E(x, y)$ under Φ' and of $E(x, x'')$ under Σ have also the common end-point. From this we see that there exists a 2-dimensional surface S of class C^1 represented as $f(u, v)$, $0\leq u\leq 1$, $0\leq v\leq 1$, which satisfies the following conditions:

$$E(x, x'):\ f(u, 0),\ 0\leq u\leq 1,$$
$$E(x, y):\ f(0, v),\ 0\leq v\leq 1,$$

for fixed u $f(u, v)$, $0\leq v\leq 1$, is the image of $E(x, y)$ under the motion Φ'' of Γ such that $x\Phi''=f(u, 0)$ and for a fixed v $f(u, v)$, $0\leq u\leq 1$, the image of $E(x, x')$ under the motion Σ' such that $x\Sigma'=f(0, v)$. Then it is easy to see that

$$(3.5) \qquad \rho(f(0,v),\ f(u,v)) = \rho(f(0,v'),\ f(u,v')), \quad 0 < u \leqq 1,\ 0 \leqq v \leqq 1.$$

Let $\lambda_{u,v}$ be the Euclidean unit vector with the same direction as a Euclidean segment $E(f(0,v),\ f(u,v))$. We have then

$$lim_{u\to 0} \frac{\rho(f(0,v),\ f(u,v))}{F(f(0,v),\ \lambda_{u,v})\, e(f(0,v),\ f(u,v))} = 1.$$

The group Γ is abelian and $lim_{u\to 0}\ \lambda_{u,v} = \mu_v\ (=\mu_{f(0,v)})$. Hence by (3.5) we get

$$(3.6) \qquad lim_{u\to 0} \frac{F(f(0,v),\ \lambda_{u,v})\, e(f(0,v),\ f(u,v))}{F(f(0,v'),\ \lambda_{u,v'})\, e(f(0,v'),\ f(u,v'))} = 1$$

for any v and v'. The tangent Minkowskian spaces at points $f(0,v)$ and $f(0,v')$ are isometric in two directions μ_v and $\mu_{v'}$. Hence we have

$$F(f(0,v),\ \mu_v) = F(f(0,v'),\ \mu_{v'}).$$

Therefore we get by (3.6)

$$lim_{u\to 0} \frac{e(f(0,v),\ f(u,v))}{e(f(0,v'),\ f(u,v'))} = 1.$$

We denote by L_u the arc $f(u,v)$, $0 \leqq v \leqq 1$, and by M_v the arc $f(u,v)$, $0 \leqq u \leqq 1$. Then L_0 and M_0 are identical with $E(x,y)$ and $E(x,x')$ respectively. Let α is a positive number and u, v and v' positive numbers such that $0 \leqq u \leqq 1$, $0 < v < v' \leqq 1$ and $\alpha = \rho(f(0,v),\ f(0,v'))/\rho(f(0,v),\ f(u,v))$. If under these conditions v' tends to v and u tends to zero, the Euclidean quadrangle $f(0,v)\,f(u,v)\,f(u,v')\,f(0,v')$ tends to the point $f(0,v)$ so as to be similar to a parallelogram.

Let P be the 2-plane determined by the segments L_0 and M_0. Then P is the tangent plane of S at $f(0,0)$. The Euclidean unit vectors $\mu_v\ (0 \leqq v \leqq 1)$ lie on P and $l_\rho(L_0) = l_\rho(L_u)$ for each u. Hence a consecutive arc L_{0+du} of L_0 lies on P and its length equals that of L_0. From this it follows that the arc L_{0+du} is a Euclidean segment parallel to L_0 and hence the Euclidean unit vectors $\mu_0(=\mu_x)$ and $\mu_1(=\mu_y)$ are parallel. Thus we see that in U_x the function $F(x,\xi)$ does not depend on the variables $x^{i'}s$ and the space is locally Minkowskian. Since the group Γ is locally compact, connected, locally connected and commutative, the last part of the theorem is clear [4]. Thus the theorem is proved.

It is also clear that, if in the theorem the indicatrices are not necessarily convex, the space is locally isometric to Minkowskian. This follows directly from the same consideration as in § 2.

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THE POINT ESTIMATION OF THE VARIANCE COMPONENTS IN RANDOM EFFECT MODEL

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1. Introduction.

In this paper we shall be concerned with the estimation of the variance components of the r-way layout of random effect model. Concerning the theory of estimation in the design of experiments the author should like at first to mention the work of R. C. Bose [2][1], where the estimation problem was fully discussed under the general linear model, which is applicable to the estimation of the treatment effects under the fixed effect model. In his work the normality of the distribution was not assumed and the arguments were solely based on the Markov Theorem.

On the other hand concerning the estimate of the variance of the error term and that of the variance components in both the fixed effect and random effect model, there have not been known so far except for the unbiasedness.

Neverthless in view of the developments of the theory of estimation as a part of the current statistical inference theory (Lehmann–Scheffé [6], Lehmann [5]), it has been felt to be needed to develop the theory in the model of design of experiments from the standpoint of the current statistical inference theory. In this connection we mention the work of Y. Washio [7], where he proved that the ordinary estimates of the parameters in the fixed effect model are the best unbiased estimates in the sense that the estimates are of uniformly minimum variance as are based on the complete sufficient statistics. Thus the problem concerning the r-way layout of the fixed effect model has been solved, and also he treated the same problem concerning the random effect model. His result, however, is restricted to the 1-way layout only.

The purpose of this paper is to treat the problem of estimation in the r-way layout of random effect model. The main difficulty in this problem lies in deriving the joint density function, and for this purpose we have to prepare with some complicated notation system in handling the variance matrix, its determinant and inverse (Theorem 4.1 and 4.2). After such cumbersome calculations, we shall come to the derivation of the joint density function (Theorem 4.3), and then we shall observe that, as is pointed out by Washio, the sufficient statistics of the family of the distribution in our concern can not be proved to be complete by the usual method appealing to the unicity of the Laplace transform. Therefore we shall prove, instead of following the line of Washio, that the estimates of the variance components ordinary used in the practice of statistical analysis are the minimum variance estimates in the sense of Bhattacharyya (Theorem 4.4). In proving it we shall appeal to the result due to Bhattacharyya [1], which enables us to prove it without verifying the lower bound of

1) Numbers in brackets refer to the references of the end of the paper.

Cramér-Rao [3] or its generalization due to Bhattacharyya is attained by the variance of the estimate.

As the arguments and the notation system are very much complicated we shall treat the special case of the 2-way layout (Section 3) as a preparatory exposition to the general case.

After treating the random effect model, there naturally arises the corresponding problem for the case of the mixed effect model, which the author wishes to discuss on another occasion.

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2. Preliminaries.

In this paper we shall be concerned with the r-way layout of random effect model whose model equation is given by the following

$$x_{t_0t_1\cdots t_r}=\mu+\sum_{k=1}^{r}\sum_{T_{i_k}\subset T_r}a_{t_{i_1}\cdots t_{i_k}}+e_{t_0t_1\cdots t_r}\quad (t_{ij}=1,2,\cdots\cdots,n_{ij},j=1,\cdots,k) \tag{2.1}$$

where μ denotes the general mean, $a_{t_{i_1}\cdots t_{i_k}}$ denotes the interraction between i_1-*th*, i_2-*th*, $\cdots\cdots$, i_k-*th* factors with the level t_{i_1}, t_{i_2}, $\cdots\cdots$, t_{i_k}, and $e_{t_0t_1\cdots t_r}$ denotes the error term. In the above equation T_r denotes the set of suffixes, $t_1, t_2,\cdots\cdots, t_r$ and T_{i_k} denotes a subset $(t_{i_1}, t_{i_2},\cdots\cdots, t_{i_k})$ of $T_r=(t_1, t_2,\cdots\cdots, t_r)$, with the relation $i_1<i_2<\cdots\cdots<i_k$. $\sum\limits_{T_{i_k}\subset T_r}$ denotes the summation for all subsets $T_{i_k}=(t_{i_1},\cdots\cdots, t_{i_k})$ of size k in $T_r=(t_1,\cdots\cdots, t_r)$, or in other words for all subsets of integers $(i_1,\cdots\cdots, i_k)$ in $(1, 2, \cdots\cdots, r)$.

We assume that μ is a constant, all $a_{t_{i_1}\cdots t_{i_k}}$ and $e_{t_0t_1\cdots t_r}$ are distributed independently to each other as normal with mean all equal to 0 and variance of $a_{t_{i_1}\cdots t_{i_k}}$ equal to $\sigma_{i_1\cdots i_k}$, the variance of $e_{t_0t_1\cdots t_r}$ all equal to σ_0.

The Kronecker product of two or any number of matrixes are defined in this paper in the way reverse to the usual ones for the convenience in handling the cumbersome notation systems, which will become clear in the course of the developments of the arguments in this paper.

Let $A=(a_{ij})$, $B=(b_{ij})$, the Kronecker product denoted by $A\otimes B$ is defined as the matrix with the (i,j)-*th* submatrix Ab_{ij} instead of $a_{ij}B$, in the usual manner. The Kronecker product of any number of matrixes is defined as the natural generalization of two matrixes, we shall write the Kronecker product of n matrixes A_1, A_2, $\cdots\cdots$, A_n, as $\prod\limits_{i=1}^{n}\otimes A_i$.

In this paper we shall make use of the well-known relations concerning the Kronecker products of two matrixes such as $(A\otimes B)(C\otimes D)=AC\otimes BD$, $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$, $(A\otimes B)'=A'\otimes B'$, and their generalizations to the products of any number of matrixes without mentioning explicitly. Throughout this paper we shall write $n\times n$ unit matrix as I_n, E_n denotes the $n\times n$ matrix with the elements all equal to 1. Let H_n be the $n\times n$ matrix with the elements all equal to zero except for the element of the first row in the first column equal to 1, and let $K_n=I_n-H_n$; namely,

$$(2.2)\qquad I_n=\begin{pmatrix}1 & & & 0\\ & 1 & & \\ & & \ddots & \\ 0 & & & 1\end{pmatrix},\quad E_n=\begin{pmatrix}1 & 1 & \cdots & 1\\ \vdots & & & \vdots\\ \vdots & & & \vdots\\ 1 & \cdots & \cdots & 1\end{pmatrix},\quad H_n=\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & 0 & & \vdots\\ \vdots & & \ddots & \vdots\\ 0 & \cdots & \cdots & 0\end{pmatrix},\quad K_n=\begin{pmatrix}0 & & & 0\\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ 0 & & & & 1\end{pmatrix}.$$

Further let T_n be defined as the orthogonal matrix with the elements of the first column all equal to $\frac{1}{\sqrt{n}}$, namely

$$(2.3)\qquad T_n=\begin{pmatrix}\frac{1}{\sqrt{n}} & \times & \times\cdots\cdots\times\\ \frac{1}{\sqrt{n}} & \times & \times\cdots\cdots\\ \vdots & \vdots & \vdots \qquad\quad \vdots\\ \frac{1}{\sqrt{n}} & \times & \times\cdots\cdots\times\end{pmatrix}.$$

Then we have easily

$$(2.4)\qquad T'_n E_n T_n = nH_n.$$

3. The case of the 2-way layout

This section is devoted to the case of the 2-way layout. Although there is no essential difference between the case of the 2-way layout and the case of the general r-way layout the notation system we need in the latter case is so cumbersome and complicated that it would be, the author feels, necessary to treat this special case for the preparatory exposition of the basic techniques in the developments of the arguments in the general case.

In this section we shall be concerned with the model equation,

$$(3.1)\qquad x_{t_0t_1t_2}=\mu+a_{t_1}+a_{t_2}+a_{t_1t_2}+e_{t_0t_1t_2}\qquad \begin{matrix}t_0=1,2,\cdots,n_0,\\ t_1=1,2,\cdots,n_1,\\ t_2=1,2,\cdots,n_2,\end{matrix}$$

where μ is a constant denoting the general mean, and $a_{t_1}, a_{t_2}, a_{t_1t_2}$ and $e_{t_0t_1t_2}$, are distributed normally with mean 0 and the variance $\sigma_1, \sigma_2, \sigma_{12}$ and σ_0, respectively, and further they are all independent to each other. Then the variance matrix of these $n_0 n_1 n_2$ variables $x_{t_0t_1t_2}$ are given by

$$(3.2)\qquad V=\begin{array}{c}\\ n_2\end{array}\overset{n_2}{\begin{pmatrix}L & M & \cdots & M\\ M & L & & \vdots\\ \vdots & & \ddots & M\\ M & \cdots & M & L\end{pmatrix}}$$

$$=\begin{pmatrix}M & \cdots & \cdots & M\\ \vdots & & & \vdots\\ \vdots & & & \vdots\\ M & \cdots & \cdots & M\end{pmatrix}+\begin{pmatrix}N & 0 & \cdots & 0\\ 0 & N & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & N\end{pmatrix}+\begin{pmatrix}P & 0 & \cdots & 0\\ 0 & P & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & P\end{pmatrix}+\begin{pmatrix}Q & 0 & \cdots & 0\\ 0 & Q & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & Q\end{pmatrix}$$

where

$$(3.3)\qquad L=\overbrace{\left(\begin{array}{c|c|c|c}
\overbrace{\begin{matrix}A & & B\\ & \ddots & \\ B & & A\end{matrix}}^{n_0} & C & \cdots\cdots & C \\ \hline
C & \begin{matrix}A & & B\\ & \ddots & \\ B & & A\end{matrix} & & \vdots \\ \hline
\vdots & & \ddots & C \\ \hline
C & \cdots\cdots & C & \begin{matrix}A & & B\\ & \ddots & \\ B & & A\end{matrix}
\end{array}\right)}^{n_1}\begin{matrix}\}n_0\\ \\ \}n_1\\ \\ \end{matrix},\qquad
M=\left(\begin{array}{c|c|c|c}
D & 0 & \cdots\cdots & 0 \\ \hline
0 & D & & \vdots \\ \hline
\vdots & & \ddots & 0 \\ \hline
0 & \cdots\cdots & 0 & D
\end{array}\right),$$

$$N=\left(\begin{array}{c|c|c}
C & \cdots\cdots\cdots & C \\ \hline
\vdots & & \vdots \\ \hline
C & \cdots\cdots\cdots & C
\end{array}\right),\qquad
P=\left(\begin{array}{c|c|c|c}
B-C-D & 0 & \cdots\cdots & 0 \\ \hline
0 & B-C-D & & \vdots \\ \hline
\vdots & & \ddots & 0 \\ \hline
0 & \cdots\cdots & 0 & B-C-D
\end{array}\right),$$

$$Q=\left(\begin{array}{c|c|c|c}
\begin{matrix}A-B & & 0\\ & \ddots & \\ 0 & & A-B\end{matrix} & 0 & \cdots\cdots & 0 \\ \hline
0 & \begin{matrix}A-B & & 0\\ & \ddots & \\ 0 & & A-B\end{matrix} & & \vdots \\ \hline
\vdots & & \ddots & 0 \\ \hline
0 & \cdots\cdots & 0 & \begin{matrix}A-B & & 0\\ & \ddots & \\ 0 & & A-B\end{matrix}
\end{array}\right),$$

and

$$(3.4)\qquad \begin{aligned}
A&=\sigma_1+\sigma_2+\sigma_{12}+\sigma_0,\\
B&=\sigma_1+\sigma_2+\sigma_{12},\\
C&=\sigma_1,\\
D&=\sigma_2.
\end{aligned}$$

This can be expressed simply in terms of the Kronecker product of the matrixes as follows.

$$(3.5)\qquad V=\sigma_1 E_{n_0}\otimes I_{n_1}\otimes E_{n_2}+\sigma_2 E_{n_0}\otimes E_{n_1}\otimes I_{n_2}+\sigma_{12}E_{n_0}\otimes I_{n_1}\otimes I_{n_2}+\sigma_0 I_{n_0}\otimes I_{n_1}\otimes I_{n_2}.$$

At first we shall evaluate the determinant of this matrix, which is equal to the determinat of the following matrix.

$$(3.6)\qquad (T_{n_0}\otimes T_{n_1}\otimes T_{n_2})'\ V(T_{n_0}\otimes T_{n_1}\otimes T_{n_2}).$$

In view of (2.4), this is equal to

$$\begin{aligned}(3.7)\qquad & n_0n_2\sigma_1 H_{n_0}\otimes I_{n_1}\otimes H_{n_2}+n_0n_1\sigma_2 H_{n_0}\otimes H_{n_1}\otimes H_{n_2}+n_0\sigma_{12}H_{n_0}\otimes I_{n_1}\otimes I_{n_2}+\sigma_0 I_{n_0}\otimes I_{n_1}\otimes I_{n_2}\\ &=n_0n_2\sigma_1 H_{n_0}\otimes(H_{n_1}+K_{n_1})\otimes H_{n_2}+n_0n_1\sigma_2 H_{n_0}\otimes H_{n_1}\otimes(H_{n_2}+K_{n_2})\\ &\quad+n_0\sigma_{12}H_{n_0}\otimes(H_{n_1}+K_{n_1})\otimes(H_{n_2}+K_{n_2})+\sigma_0(H_{n_0}+K_{n_0})\otimes(H_{n_1}+K_{n_1})\otimes(H_{n_2}+K_{n_2})\\ &=(n_0n_2\sigma_1+n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)H_{n_0}\otimes H_{n_1}\otimes H_{n_2}+(n_0n_2\sigma_1+n_0\sigma_{12}+\sigma_0)H_{n_0}\otimes K_{n_1}\otimes H_{n_2}\\ &\quad+(n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)H_{n_0}\otimes H_{n_1}\otimes K_{n_2}+(n_0\sigma_{12}+\sigma_0)H_{n_0}\otimes K_{n_1}\otimes K_{n_2}\\ &\quad+\sigma_0\{K_{n_0}\otimes H_{n_1}\otimes H_{n_2}+K_{n_0}\otimes K_{n_1}\otimes H_{n_2}+K_{n_0}\otimes H_{n_1}\otimes K_{n_2}+K_{n_0}\otimes K_{n_1}\otimes K_{n_2}\}.\end{aligned}$$

Thus the matrix (3.6) is expressed as the linear form of eight matrixes, and as all of them are diagonal, this matrix is also diagonal, and any two matrixes have no non-zero element in common. This fact leads to the evaluation of the determinant as follows,

$$\begin{aligned}(3.8)\qquad |V|=&(n_0n_2\sigma_1+n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)\,(n_0n_2\sigma_1+n_0\sigma_{12}+\sigma_0)^{(n_1-1)}(n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)^{(n_2-1)}\\ &\cdot(n_0\sigma_{12}+\sigma_0)^{(n_1-1)(n_2-1)}\sigma_0^{n_1n_2(n_0-1)},\end{aligned}$$

or by writing

$$\begin{aligned}(3.9)\qquad &\theta_0=\sigma_0,\\ &\theta_{12}=n_0\sigma_{12}+\sigma_0,\\ &\theta_1=n_0n_2\sigma_1+n_0\sigma_{12}+\sigma_0,\\ &\theta_2=n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0,\\ &\theta_E=n_0n_2\sigma_1+n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0,\end{aligned}$$

we have finally

$$(3.10)\qquad |V|=\theta_E\cdot\theta_1^{(n_1-1)}\theta_2^{(n_2-1)}\theta_{12}^{(n_1-1)(n_2-1)}\theta_0^{n_1n_2(n_0-1)}.$$

Now let us find out the inverse matrix of the variance matrix (3.5). The variance matrix is given as the linear form of four matrixes, and anticipating its inverse to be a linear form of these four and of $E_{n_0}\otimes E_{n_1}\otimes E_{n_2}$, we have, after a simple calculations,

$$\begin{aligned}(3.11)\qquad &[\sigma_1 E_{n_0}\otimes I_{n_1}\otimes E_{n_2}+\sigma_2 E_{n_0}\otimes E_{n_1}\otimes I_{n_2}+\sigma_{12}E_{n_0}\otimes I_{n_1}\otimes I_{n_2}+\sigma_0 I_{n_0}\otimes I_{n_1}\otimes I_{n_2}]\\ &\quad\cdot[X_E E_{n_0}\otimes E_{n_1}\otimes E_{n_2}+X_1 E_{n_0}\otimes I_{n_1}\otimes E_{n_2}+X_2 E_{n_0}\otimes E_{n_1}\otimes I_{n_2}\\ &\quad+X_{12}E_{n_0}\otimes I_{n_1}\otimes I_{n_2}+X_0 I_{n_0}\otimes I_{n_1}\otimes I_{n_2}]\\ &=E_{n_0}\otimes E_{n_1}\otimes E_{n_2}[(n_0n_2\sigma_1+n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)X_E+n_0\sigma_2X_1+n_0\sigma_1X_2]\\ &\quad+E_{n_0}\otimes I_{n_1}\otimes E_{n_2}[(n_0n_2\sigma_1+n_0\sigma_{12}+\sigma_0)X_1+n_0\sigma_1X_{12}+\sigma_1X_0]\\ &\quad+E_{n_0}\otimes E_{n_1}\otimes I_{n_2}[(n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)X_2+n_0\sigma_2X_{12}+\sigma_2X_0]\end{aligned}$$

$$+E_{n_0}\otimes I_{n_1}\otimes I_{n_2}[(n_0\sigma_{12}+\sigma_0)X_{12}+\sigma_{12}X_0]$$
$$+I_{n_0}\otimes I_{n_1}\otimes I_{n_2}X_0\sigma_0.$$

In order to have the second matrix which is a linear form of five matrixes to be the inverse of the first, the product of these two should be the unit matrix, for which we should have

$$\begin{aligned}
&(n_0n_2\sigma_1+n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)X_E+n_0\sigma_2X_1+n_0\sigma_1X_2=0,\\
&(n_0n_2\sigma_1+n_0\sigma_{12}+\sigma_0)X_1+n_0\sigma_1X_{12}+\sigma_1X_0=0,\\
(3.12)\quad &(n_0n_1\sigma_2+n_0\sigma_{12}+\sigma_0)X_2+n_0\sigma_2X_{12}+\sigma_2X_0=0,\\
&(n_0\sigma_{12}+\sigma_0)X_{12}+\sigma_{12}X_0=0,\\
&\sigma_0X_0=1.
\end{aligned}$$

The solution of these linear equations is given by

$$\begin{aligned}
&X_0=\frac{1}{\theta_0},\\
&X_{12}=\frac{1}{n_0}\left(\frac{1}{\theta_{12}}-\frac{1}{\theta_0}\right),\\
(3.13)\quad &X_1=\frac{1}{n_0n_2}\left(\frac{1}{\theta_1}-\frac{1}{\theta_{12}}\right),\\
&X_2=\frac{1}{n_0n_1}\left(\frac{1}{\theta_2}-\frac{1}{\theta_{12}}\right),\\
&X_E=\frac{1}{n_0n_1n_2}\left(\frac{1}{\theta_{12}}-\frac{1}{\theta_1}-\frac{1}{\theta_2}+\frac{1}{\theta_E}\right).
\end{aligned}$$

Thus we have obtained the determinant and the inverse of the variance matrix, which enables us to give the joint density function of all the $n_0n_1n_2$ variables in our concern. By noting the relations

$$(3.14)\quad \mathfrak{X}_n'E_n\mathfrak{X}_n=\left(\sum_{i=1}^{n}x_i\right)^2,\quad \mathfrak{X}_n'I_n\mathfrak{X}_n=\sum_{i=1}^{n}x_i^2$$

where $\mathfrak{X}_n$ is any n-dimensional vector $\mathfrak{X}_n'=(x_1, x_2, \cdots\cdots, x_n)$, and by writing

$$(3.15)\quad u_{t_0t_1t_2}=x_{t_0t_1t_2}-\mu,$$

our joint density function is given by the following

$$(3.16)\quad f(\mathbf{X})=\left(\frac{1}{\sqrt{2\pi}}\right)^{n_0n_1n_2}(|V|)^{-1/2}\,exp\left(-\frac{1}{2}S\right),$$

where

$$\begin{aligned}
(3.17)\quad S=&X_E\Big(\sum_{t_0}\sum_{t_1}\sum_{t_2}u_{t_0t_1t_2}\Big)^2+X_1\sum_{t_1}\Big(\sum_{t_0}\sum_{t_2}u_{t_0t_1t_2}\Big)^2+X_2\sum_{t_2}\Big(\sum_{t_0}\sum_{t_1}u_{t_0t_1t_2}\Big)^2\\
&+X_{12}\sum_{t_1}\sum_{t_2}\Big(\sum_{t_0}u_{t_0t_1t_2}\Big)^2+X_0\sum_{t_0}\sum_{t_1}\sum_{t_2}u_{t_0t_1t_2}^2,
\end{aligned}$$

where X_E, X_1, X_2, X_{12}, and X_0 are given by (3.12).

After a simple modification, we have finally

$$(3.18)\quad f(\mathbf{X})=\mathrm{K}\,\theta_E^{-1/2}\,\theta_1^{-(n_1-1)/2}\,\theta_2^{-(n_2-1)/2}\,\theta_{12}^{-(n_1-1)(n_2-1)/2}\,\theta_0^{-n_1n_2(n_0-1)/2}$$

$$\cdot exp\Big[-\frac{1}{2}\Big\{\frac{1}{\theta_0}\sum_{t_0}\sum_{t_1}\sum_{t_2}(x_{t_0t_1t_2}-\bar{x}_{\cdot t_1t_2})^2+\frac{n_1n_2}{\theta_1}\sum_{t_1}(\bar{x}_{\cdot t_1\cdot}-\bar{x}_{\cdots})^2$$
$$+\frac{n_0n_1}{\theta_2}\sum_{t_2}(\bar{x}_{\cdot\cdot t_2}-\bar{x}_{\cdots})^2+\frac{n_0}{\theta_{12}}\sum_{t_1}\sum_{t_2}(\bar{x}_{\cdot t_1t_2}-\bar{x}_{\cdot t_1\cdot}-\bar{x}_{\cdot\cdot t_2}+\bar{x}_{\cdots})^2+\frac{n_0n_1n_2}{\theta_E}(\bar{x}_{\cdots}-\mu)^2\Big\}\Big]$$

where K is a constant independent of the parameters in our concern.

We have the family of distributions whose parameter space is written explicitly as

$$(3.19)\quad \Omega=\left(\begin{array}{ll} 0\leqq\theta_0<\infty, & \theta_E=\theta_1+\theta_2-\theta_{12}, \\ \theta_0\leqq\theta_{12}<\infty, & \\ \theta_{12}\leqq\theta_1<\infty, & -\infty<\mu<\infty, \\ \theta_{12}\leqq\theta_2<\infty, & \end{array}\right),$$

and whose sufficient statistics are given by the following five statistics

$$(3.20)\quad \begin{aligned} S_0&=\sum_{t_0}\sum_{t_1}\sum_{t_2}(x_{t_0t_1t_2}-\bar{x}_{\cdot t_1t_2})^2,\\ S_1&=n_0n_2\sum_{t_1}(\bar{x}_{\cdot t_1\cdot}-\bar{x}_{\cdots})^2,\\ S_2&=n_0n_1\sum_{t_2}(\bar{x}_{\cdot\cdot t_2}-\bar{x}_{\cdots})^2,\\ S_{12}&=n_0\sum_{t_1}\sum_{t_2}(\bar{x}_{\cdot t_1t_2}-\bar{x}_{\cdot t_1\cdot}-\bar{x}_{\cdot\cdot t_2}+\bar{x}_{\cdots})^2,\\ &\bar{x}_{\cdots}. \end{aligned}$$

If the family of distribution of these sufficient statistics is complete, the theory of estimation tells us that the usual estimates are the unique unbiased minimum variance estimates of the variance components σ_0, σ_1, σ_2, σ_{12} and the general mean μ (Lehmann [5]). As already pointed out by Washio, we have not so far been able to conclude whether it is complete or not, and we shall appeal to the notion of the minimum variance estimate due to Bhattacharyya, and in this connection we shall make use of the result due to him [1] (c. f. section 6 of Chapter I in his paper).

In view of (3.18), (3.20) we have

$$(3.21)\quad \ln f=\mathrm{K}'-\frac{1}{2}\Big\{\ln\theta_E+(n_1-1)\ln\theta_1+(n_2-1)\ln\theta_2+(n_1-1)(n_2-1)\ln\theta_{12}+$$
$$+n_1n_2(n_0-1)\ln\theta_0\Big\}-\frac{1}{2}\Big\{\frac{S_0}{\theta_0}+\frac{S_1}{\theta_1}+\frac{S_2}{\theta_2}+\frac{S_{12}}{\theta_{12}}+\frac{n_0n_1n_2(\bar{x}_{\cdots}-\mu)^2}{\theta_E}\Big\}$$

Hence we have

$$(3.22)\quad \begin{aligned} &\frac{2\theta_0^2}{n_1n_2(n_0-1)}\frac{\partial f}{\partial\theta_0}=\left(\frac{S_0}{n_1n_2(n_0-1)}-\theta_0\right)f,\\ &\frac{2\theta_1^2}{(n_1-1)}\frac{\partial f}{\partial\theta_1}-\frac{\theta_1^2}{n_0n_1n_2(n_1-1)}\frac{\partial^2 f}{\partial\mu^2}=\left(\frac{S_1}{(n_1-1)}-\theta_1\right)f,\\ &\frac{2\theta_1^2}{(n_2-1)}\frac{\partial f}{\partial\theta_2}-\frac{\theta_2^2}{n_0n_1n_2(n_2-1)}\frac{\partial^2 f}{\partial\mu^2}=\left(\frac{S_2}{(n_2-1)}-\theta_2\right)f, \end{aligned}$$

$$-\frac{2\theta_{12}^2}{(n_1-1)(n_2-1)}\frac{\partial f}{\partial\theta_{12}}+\frac{\theta_{12}^2}{n_0n_1n_2(n_1-1)(n_2-1)}\frac{\partial^2 f}{\partial\mu^2}=\left(\frac{S_{12}}{(n_1-1)(n_2-1)}-\theta_{12}\right)f,$$

$$\frac{\theta_E}{n_0n_1n_2}\frac{\partial f}{\partial\mu}=(\bar{x}\ldots-\mu)f.$$

This fact and the result in Bhattacharyya [1] yield us that the minimum variance estimates of $\sigma_0, \sigma_1, \sigma_2, \sigma_{12}$ and μ is given by $\frac{S_0}{n_1n_2(n_0-1)}$, $\frac{S_1}{n_0n_2(n_1-1)}-\frac{S_{12}}{n_0n_2(n_1-1)(n_2-1)}$, $\frac{S_2}{n_0n_1(n_2-1)}-\frac{S_{12}}{n_0n_1(n_1-1)(n_2-1)}$, $\frac{S_{12}}{n_0(n_1-1)(n_2-1)}-\frac{S_0}{n_0n_1n_2(n_0-1)}$, $\bar{x}\ldots$.

4. The case of the r-way layout.

4.1 *The determinant of the variance matrix.* In this section we shall give the results and the proofs in the case of the r-way layout with the model given in (2.1) under the assumptions stated in the beginning of section 2.

Corresponding to (3.4), we have the expression of the variance matrix in terms of the Kronecker products as follows

$$(4.1)\quad V=\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots\cdots i_k}E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\cdots i_k}}\times I_{n_j}^{\delta^j_{i_1\cdots i_k}}\right)+\sigma_0 I_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r},$$

where $\delta^j_{i_1\cdots\cdots i_k}$ is a sort of generalization of the Kronecker's delta which is

$$(4.2)\quad \delta^j_{i_1\cdots\cdots i_k}=\begin{cases}1 & \text{if } j \text{ is equal to either of } (i_1, i_2, \cdots\cdots, i_k)\\ 0 & \text{otherwise.}\end{cases}$$

and E^0 of a matrix E is defined to be the unit matrix I. The reason for this expression is clear.

Throughout this paper the notations such as T_α, S_β, I_k etc. mean a set of integers $(t_1, t_2, \cdots\cdots, t_\alpha)$, $(s_1, s_2, \cdots\cdots, s_\beta)$, $(i_1, i_2, \cdots\cdots, i_k)$ etc. and $R=(1, 2, 3, \cdots\cdots, r)$, and the summations such as $\sum_{A\subset B} a_A$, $\sum_{\substack{A\subset B\\ A\supset C}} a_A$, where A, B, C are such sets of integers as stated above, mean the sum of all numbers a_A's having A as the suffixes which are included in B, or included in B and including C, respectively.

For the developments of the arguments in this section we have to prepare with a number of notations as follows.

DEFINITION 4.1.

$$(4.3)\quad A_{(t_1\cdots\cdots t_\alpha)}=\sum_{k=\alpha}^{r}\sum_{\substack{I_k\supset T_\alpha\\ I_k\subset R}}\sigma_{i_1\cdots\cdots i_k}\prod_{j=0}^{r}n^{1-\delta^j_{i_1\cdots\cdots i_k}},$$

$$(4.4)\quad A^{(s_1\cdots\cdots s_\beta)}_{(t_1\cdots\cdots t_\alpha)}=\sum_{k=\alpha}^{r-\beta}\sum_{\substack{I_k\supset T_\alpha\\ I_k\subset R-S_\beta}}\sigma_{i_1\cdots\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots\cdots i_k s_1\cdots\cdots s_\beta}},$$

$$(4.5)\quad A=\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots\cdots i_k}},$$

$$(4.6)\qquad A_{(t_1\cdots\cdots t_\alpha)}+\sigma_0=B_{(t_1\cdots\cdots t_\alpha)},$$

$$(4.7)\qquad A+\sigma_0=B.$$

Now we shall evalute at first the determinant of the variance matrix (4.1)

THEOREM 4.1. *The determinant* $|V|$ *of the variance matrix* V *of* (4.1) *is given in the notation of* (4.6) *and* (4.7) *as follows*

$$(4.8)\qquad |V|=B\cdot\prod_{k=1}^{r}\prod_{I_k\subset R}\left\{B_{(i_1\cdots\cdots i_k)}\right\}^{(n_{i_1}-1)(n_{i_2}-1)\cdots(n_{i_k}-1)}\sigma_0^{(n_0-1)n_1\cdots\cdots n_r}.$$

PROOF. Let us at first transform this matrix by the orthogonal matrix which is the Kronecker product of the matrixes T_{n_i} defined in (2.3), and we have

$$\begin{aligned}(4.9)\qquad &(T_{n_0}\otimes T_{n_1}\otimes\cdots\cdots\otimes T_{n_r})'V(T_{n_0}\otimes T_{n_1}\otimes\cdots\cdots\otimes T_{n_r})\\ &=\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}H_{n_0}\otimes\prod_{t=1}^{r}\otimes\left(H_{n_t}^{1-\delta^t_{i_1\cdots i_k}}\times I_{n_t}^{\delta^t_{i_1\cdots i_k}}\right)+\sigma_0 I_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}\\ &=\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}H_{n_0}\otimes\prod_{t=1}^{r}\otimes\left(H_{n_t}^{1-\delta^t_{i_1\cdots\cdots i_k}}\times(H_{n_t}+K_{n_t})^{\delta^t_{i_1\cdots i_k}}\right)\\ &\quad+\sigma_0\prod_{j=1}^{r}\otimes(H_{n_j}+K_{n_j})\\ &=\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}H_{n_0}\otimes\prod_{t=1}^{r}\otimes(H_{n_t}+\delta^t_{i_1\cdots i_k}K_{n_t})+\sigma_0\prod_{j=1}^{r}\otimes(H_{n_j}+K_{n_j}).\end{aligned}$$

This matrix is also a linear form of the matrixes of the type

$$(4.10)\qquad H_{n_0}\otimes A_1\otimes A_2\otimes\cdots\cdots\otimes A_r,$$

where

$$(4.11)\qquad A_i=H_{n_i}\ \text{or}\ H_{n_i}+K_{n_i}\equiv I_{n_i}.$$

There are 2^r different matrixes of this type, which are all diagonal, and hence the matrix (4.9) is diagonal. The product of any two matrixes of this type is the null matrix, and the matrix (4.9) itself is a nonsingular matrix. Therefore the determinant is equal to the product of $n_0n_1\cdots\cdots n_r$ numbers each of which is equal to either of the coefficient of 2^r different matrixes in the linear form (4.9). In this product the coefficient of $H_{n_0}\otimes(H_{n_1}+K_{n_1})\otimes H_{n_2}\otimes\cdots\cdots\otimes H_{n_r}$, for instance, appears exactly n_1-1 times which is equal to the rank of this matrix, and this coefficient is equal to $B_{(1)}^{(n_1-1)}$.

Thus we have

$$\begin{aligned}(4.12)\qquad |V|=&\left[\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}+\sigma_0\right]\\ &\cdot\prod_{k=1}^{r}\prod_{I_k\subset R}\left\{\sum_{p=k}^{r}\sum_{\substack{L_p\supset I_k\\ L_p\subset R}}\sigma_{l_1\cdots\cdots l_p}\prod_{j=1}^{r}n_j^{1-\delta^j_{l_1\cdots l_p}}+\sigma_0\right\}^{(n_{i_1}-1)(n_{i_2}-1)\cdots(n_{i_k}-1)}\cdot\ \sigma_0^{(n_0-1)n_1\cdots n_r},\end{aligned}$$

which is equal to (4.8) from the definition 4.1.

4.2 *The inverse of the variance matrix.* Before finding out the inverse of the

variance matrix, we need to consider some relations between the notations defined in the definition 4.1. At first we observe the recurrence relation of $A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_\beta)}$.

LEMMA 4.1.

$$(4.13)\quad A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_\beta)}=\frac{1}{n_{s_\beta}}\left[A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_{\beta\text{-}1})}-A_{(t_1\cdots t_\alpha s_\beta)}^{(s_1\cdots s_{\beta\text{-}1})}\right].$$

PROOF.

$$(4.14)\quad A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_\beta)}=\sum_{k=\alpha}^{r-\beta}\sum_{\substack{I_k\supset T_\alpha\\ I_k\subset R-S_\beta}}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_\beta}}$$

$$=\sum_{k=\alpha}^{r-\beta+1}\sum_{\substack{I_k\supset T_\alpha\\ I_k\subset R-S_\beta}}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_\beta}}-\sum_{k=\alpha+1}^{r-\beta+1}\sum_{\substack{I_k\supset(T_\alpha,S_\beta)\\ I_k\subset R-S_\beta}}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_\beta}}$$

$$=\frac{1}{n_{s_\beta}}\left[\sum_{k=\alpha}^{r-\beta+1}\sum_{\substack{I_k\supset T_\alpha\\ I_k\subset R-S_{\beta\text{-}1}}}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_{\beta\text{-}1}}}-\sum_{k=\alpha+1}^{r-\beta+1}\sum_{\substack{I_k\supset(T_\alpha,S_\beta)\\ I_k\subset R-S_{\beta\text{-}1}}}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_{\beta\text{-}1}}}\right]$$

$$=\frac{1}{n_{s_\beta}}\left[A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_{\beta\text{-}1})}-A_{(t_1\cdots t_\alpha s_\beta)}^{(s_1\cdots s_{\beta\text{-}1})}\right].$$

This lemma enables us to express $A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_\beta)}$ in terms of $A_{(t_1\cdots t_\alpha l_1\cdots l_p)}$, which is given by

LEMMA 4.2.

$$(4.15)\quad A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_\beta)}=\frac{1}{n_{s_1}\cdots n_{s_\beta}}\sum_{p=0}^{\beta}\sum_{L_p\subset S_\beta}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)}.$$

PROOF.

We shall give the proof by making use of the mathematical induction in β. In case $\beta=1$, we have

$$(4.16)\quad A_{(t_1\cdots t_\alpha)}^{(s_1)}=\frac{1}{n_{s_1}}\left[A_{(t_1\cdots t_\alpha)}-A_{(t_1\cdots t_\alpha s_1)}\right]$$

$$=\frac{1}{n_{s_1}}\sum_{p=0}^{1}\sum_{L_p\subset S_1}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)}.$$

Then assuming (4.15) to be valid in case $\beta=h$ i. e.,

$$(4.17)\quad A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_h)}=\frac{1}{n_{s_1}\cdots n_{s_h}}\sum_{p=0}^{h}\sum_{L_p\subset S_h}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)}$$

we shall prove this is also valid in case $\beta=h+1$, which is given by

$$(4.18)\quad A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_{h+1})}=\frac{1}{n_{s_{h+1}}}\left[A_{(t_1\cdots t_\alpha)}^{(s_1\cdots s_h)}-A_{(t_1\cdots t_\alpha s_{h+1})}^{(s_1\cdots s_h)}\right]$$

$$=\frac{1}{n_{s_{h+1}}}\left[\frac{1}{n_{s_1}\cdots n_{s_h}}\left\{\sum_{p=0}^{h}\sum_{L_p\subset S_h}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)}-\sum_{p=0}^{h}\sum_{L_p\subset S_h}(-1)^p A_{(t_1\cdots t_\alpha s_{h+1} l_1\cdots l_p)}\right\}\right]$$

$$=\frac{1}{n_{s_1}\cdots n_{s_{h+1}}}\left[\sum_{p=0}^{h}\sum_{L_p\subset S_h}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)}+\sum_{p=0}^{h+1}\sum_{\substack{L_p\subset S_{h+1}\\ L_p\ni s_{h+1}}}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)}\right]$$

$$= \frac{1}{n_{s_1}\cdots n_{s_{h+1}}}\sum_{p=0}^{h+1}\sum_{L_p\subset S_{h+1}}(-1)^p A_{(t_1\cdots t_\alpha l_1\cdots l_p)}.$$

Now let us turn to the inversion of the variance matrix. The arguments follow the similar line to that of the 2-way layout.

THEOREM 4.2. *The inverse of the variance matrix (2.1) is given by*

$$(4.19)\quad X_E E_{n_0}\otimes E_{n_1}\otimes\cdots\cdots\otimes E_{n_r}+\sum_{k=1}^{r}\sum_{I_k\subset R}X_{i_1\cdots i_k}E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\cdots i_k}}\times I_{n_j}^{\delta^j_{i_1\cdots i_k}}\right)$$
$$+\,X_0 I_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r},$$

where

$$(4.20)\quad X_0=\frac{1}{\sigma_0},$$

$$(4.21)\quad X_{123\cdots r}=\frac{1}{n_0}\left[\frac{1}{B_{(12\cdots r)}}-\frac{1}{\sigma_0}\right],$$

$$(4.22)\quad X_{i_1 i_2\cdots i_k}=\frac{1}{\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}}\left[\frac{(-1)^{r-k}}{B_{(12\cdots r)}}+\sum_{\alpha=0}^{r-k-1}\sum_{S_\alpha\subset R-I_k}\frac{(-1)^\alpha}{B_{(i_1\cdots i_k s_1\cdots s_\alpha)}}\right](I_k\subset R,\ k=1,2,\cdots,r-1),$$

$$(4.23)\quad X_E=\frac{1}{\prod_{j=0}^{r}n_j}\left[\frac{(-1)^r}{B_{(12\cdots r)}}+\frac{1}{B}+\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c}{B_{(i_1\cdots i_c)}}\right].$$

PROOF. As we have done in case of the 2-way layout, anticipating the inverse to be the form of (4.19), we seek for the condition that (4.19) is actually the inverse. The product of the variance matrix (4.1) and the matrix (4.19) is

$$(4.24)\quad E_{n_0}\otimes E_{n_1}\otimes\cdots\cdots\otimes E_{n_r}\left[X_E\left\{\sum_{k=1}^{r}\sum_{I_k\subset R}\sigma_{i_1\cdots i_k}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}+\sigma_0\right\}\right.$$
$$\left.+\sum_{k=1}^{r}\sum_{I_k\subset R}X_{i_1\cdots i_k}\left\{\sum_{l=1}^{r-k}\sum_{T_l\subset R-I_k}\sigma_{t_1\cdots t_l}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k t_1\cdots t_l}}\right\}\right]$$
$$+\sum_{k=1}^{r-1}\sum_{I_k\subset R}E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\cdots i_k}}\times I_{n_j}^{\delta^j_{i_1\cdots i_k}}\right)$$
$$\cdot\left[\sum_{l=k}^{r}\sum_{T_l\supset I_k}X_{t_1\cdots t_l}\left\{\sum_{m=k}^{r-l+k}\sum_{\substack{S_m\supset I_k\\ S_m\subset R-(T_l-I_k)}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r}n_j^{1-\delta^j_{t_1\cdots t_l s_1\cdots s_m}}\right\}+X_0\sigma_{i_1\cdots i_k}\right]$$
$$+\,E_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}[X_{12\cdots r}(n_0\sigma_{12\cdots r}+\sigma_0)+X_0\sigma_{12\cdots r}]$$
$$+\,I_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}\cdot X_0\sigma_0$$
$$=E_{n_0}\otimes E_{n_1}\otimes\cdots\cdots\otimes E_{n_r}\left[X_E(A+\sigma_0)+\sum_{k=1}^{r}\sum_{I_k\subset R}X_{i_1\cdots i_k}A^{(i_1\cdots i_k)}\right]$$
$$+\sum_{k=1}^{r-1}\sum_{I_k\subset R}E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\cdots i_k}}\times I_{n_j}^{\delta^j_{i_1\cdots i_k}}\right)\left[X_{i_1\cdots i_k}\left\{\sum_{m=k}^{r}\sum_{\substack{S_m\supset I_k\\ S_m\subset R}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_m}}+\sigma_0\right\}\right.$$
$$\left.+\sum_{l=k+1}^{r}\sum_{T_l\supsetneqq I_k}X_{t_1\cdots t_l}\left\{\sum_{m=k}^{r-l+k}\sum_{\substack{S_m\supset I_k\\ S_m\subset R-(T_l-I_k)}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r}n_j^{1-\delta^j_{t_1\cdots t_l s_1\cdots s_m}}\right\}+X_0\sigma_{i_1\cdots i_k}\right]$$

$$+ E_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}[X_{12}\ \ ,B_{(12\cdot r)}+X_0\sigma_{12\cdots r}]$$
$$+ I_{n_0}\otimes I_{n_1}\otimes\cdots\cdots\otimes I_{n_r}X_0\sigma_0$$

The first term is equal to

$$(4.25)\quad E_{n_0}\otimes E_{n_1}\otimes\cdots\cdots\otimes E_{n_r}\left[X_E(A+\sigma_0)+\sum_{k=1}^{r}\sum_{I_k\subset R}X_{i_1\cdots i_k}A^{(i_1\cdots i_k)}\right]$$
$$= E_{n_0}\otimes E_{n_1}\otimes\cdots\cdots\otimes E_{n_r}\left[X_E B+\sum_{k=1}^{r}\sum_{I_k\subset R}X_{i_1\cdots i_k}A^{(i_1\cdots i_k)}\right]$$

The second term is equal to

$$(4.26)\quad \sum_{k=1}^{r-1}\sum_{I_k\subset R}E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1}\ \ i_k}\times I_{n_j}^{\delta^j_{i_1}\ \ i_k}\right)$$
$$\cdot\left[X_{i_1\cdots i_k}\left\{\sum_{m=k}^{r}\sum_{\substack{S_m\supset I_k\\ S_m\subset R}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k s_1\cdots s_m}}+\sigma_0\right\}\right.$$
$$\left.+\sum_{l=k+1}^{r}\sum_{T_l\supsetneqq I_k}X_{t_1\cdots t_l}\left\{\sum_{m=k}^{r-l+k}\sum_{\substack{S_m\supset I_k\\ S_m\subset R-(T_l-I_k)}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r}n_j^{1-\delta^j_{t_1\cdots t_l s_1\cdots s_m}}\right\}+X_0\sigma_{i_1\cdots i_k}\right]$$
$$=\sum_{k=1}^{r-1}\sum_{I_k\subset R}E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1}\ \ i_k}\times I_{n_j}^{\delta^j_{i_1}\ \ i_k}\right)$$
$$\cdot\left[X_{i_1\cdots i_k}\left\{\sum_{m=k}^{r}\sum_{\substack{S_m\supset I_k\\ S_m\subset R}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r}n_j^{1-\delta^j_{s_1\cdots s_m}}+\sigma_0\right\}\right.$$
$$\left.+\sum_{l=1}^{r-k}\sum_{T_l\subset R-I_k}X_{i_1\cdots i_k t_1\cdots t_l}\left\{\sum_{m=k}^{r-l}\sum_{\substack{S_m\supset I_k\\ S_m\subset R-T_l}}\sigma_{s_1\cdots s_m}\prod_{j=0}^{r}n_j^{1-\delta^j_{t_1\cdots t_l s_1\cdots s_m}}\right\}+X_0\sigma_{i_1\cdots i_k}\right]$$
$$=\sum_{k=1}^{r-1}\sum_{I_k\subset R}E_{n_0}\otimes\prod_{j=1}^{r}\otimes\left(E_{n_j}^{1-\delta^j_{i_1\cdots i_k}}\times I_{n_j}^{\delta^j_{i_1\cdots i_k}}\right)$$
$$\cdot\left[X_{i_1\cdots i_k}B_{(i_1\cdots i_k)}+\sum_{l=1}^{r-k}\sum_{T_l\subset R-I_k}X_{i_1\cdots i_k t_1\cdots t_l}A^{(t_1\cdots t_l)}_{(i_1\cdots i_k)}+X_0A^{(t_1\cdots t_{r-k})}_{(i_1\cdots i_k)}/n_0\right].$$

Thus the condition is expressed by the following equations, which are the generalization of (3.11)

$$(4.27)\quad X_0\sigma_0=1,$$

$$(4.28)\quad X_{12\cdots r}B_{(12\cdots r)}=-X_0\sigma_{12\cdots r},$$

$$(4.29)\quad X_{i_1\cdots i_k}B_{(i_1\cdots i_k)}=-\sum_{\beta=1}^{r-k}\sum_{T_\beta\subset R-I_k}X_{i_1\cdots i_k t_1\cdots t_\beta}A^{(t_1\cdots t_\beta)}_{(i_1\cdots i_k)}-X_0A^{(t_1\cdots t_{r-k})}_{(i_1\cdots i_k)}/n_0,$$

$$(4.30)\quad X_E B=-\sum_{k=1}^{r}\sum_{I_kR}X_{i_1\cdots i_k}A^{(i_1\cdots i_k)}.$$

The proof of this theorem is completed, it is obvious, by proving the following:

LEMMA 4.3. *The solutions of the equations* (4.27),$\cdots\cdots$, (4.30) *are given by* (4.20),$\cdots\cdots$, (4.23).

PROOF. (4.20) comes from (4.27) directly and (4.21) comes from (4.20) and (4.28). (4.22) is obtained by mathematical induction in k and (4.20) and (4.21), which

is as follows. At the first stage, we shall prove (4.22) holds true for all $(i_1, i_2 \cdots\cdots, i_k) = I_k \subset R$ when $k=r-1$. Then we shall prove, assuming that this holds true for all $I_k \subset R$ when $k=r-q,\ r-q+1, \cdots\cdots, r-1$, this also holds true for all $I_k \subset R$ when $k=r-q-1$.

The equations to be solved in the first stage is

$$
\begin{aligned}
(4.31)\quad & X_{i_1\cdots i_{r-1}} B_{(i_1\cdots i_{r-1})} \\
&= -X_{12\cdots r} A^{(i_r)}_{(i_1\cdots i_{r-1})} - X_0 A^{(i_r)}_{(i_1\cdots i_{r-1})}/n_0 \\
&= -\left[X_{12\cdots r} + \frac{X_0}{n_0}\right] A^{(i_r)}_{(i_1\cdots i_{r-1})} \\
&= -\left[\frac{1}{n_0}\left\{\frac{1}{B_{(12\cdots r)}} - \frac{1}{\sigma_0}\right\} + \frac{1}{n_0\sigma_0}\right] \frac{1}{n_{i_r}} \left\{A_{(i_1\cdots i_{r-1})} - A_{(12\cdots r)}\right\} \\
&= -\frac{1}{n_0 n_{i_r}} \left\{\frac{B_{(i_1\cdots i_{r-1})} - B_{(12\cdots r)}}{B_{(12\cdots r)}}\right\} \qquad (I_{r-1} \subset R).
\end{aligned}
$$

Hence we have

$$
(4.32)\quad X_{i_1\cdots i_{r-1}} = \frac{1}{n_0 n_{i_r}} \left\{\frac{-1}{B_{(12\cdots r)}} + \frac{1}{B_{(i_1\cdots i_{r-1})}}\right\} \qquad (I_{r-1} \subset R).
$$

which completes the first stage.

For the proof of the second stage, at first we observe in view of the assumptions for the mathematical induction,

$$
\begin{aligned}
(4.33)\quad & X_{i_1\cdots i_{r-q-1}} B_{(i_1\cdots i_{r-q-1})} \\
&= -\sum_{\beta=1}^{q+1} \sum_{T\beta \subset R - Ir\text{-}q\text{-}1} X_{i_1\cdots i_{r-q-1} t_1 \cdots t_\beta} A^{(t_1\cdots t_\beta)}_{(i_1\cdots i_{r-q-1})} - \frac{1}{n_0\sigma_0}\left[n_0\sigma_0 X_{12\cdots r} + 1\right] A^{(t_1\cdots t_{q+1})}_{(i_1\cdots i_{r-q-1})} \\
&= \frac{1}{N_{r-q-1}} \left[-\sum_{\beta=1}^{q+1} \sum_{T\beta \subset R - Ir\text{-}q\text{-}1} \left\{\frac{(-1)^{q+1-\beta}}{B_{(12\cdots r)}} + \sum_{\alpha=0}^{q-\beta} \sum_{S\alpha \subset R - (Ir\text{-}q\text{-}1 \cup T\beta)} \frac{(-1)^\alpha}{B_{(i_1\cdots i_{r-q-1} t_1 \cdots t_\beta s_1 \cdots s_\alpha)}}\right\} \right. \\
&\quad \left. \cdot \sum_{p=0}^{\beta} \sum_{Lp \subset T\beta} (-1)^p A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)} - \frac{1}{B_{(12\cdots r)}} \sum_{p=0}^{q+1} \sum_{Lp \subset R - Ir\text{-}q\text{-}1} (-1)^p A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)} \right],
\end{aligned}
$$

where

$$
(4.34)\quad N_{r-q-1} = \prod_{j}^{r} n_j^{1-\delta^j_{i_1\cdots i_{r-q-1}}},
$$

Now, putting $C_{(p)} = A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)}$, the coefficient to $\frac{1}{B_{(12\cdots r)}}$ is given by

$$
\begin{aligned}
(4.35)\quad & -\sum_{\beta=1}^{q} \sum_{T\beta \subset R - I_{r-q-1}} \left\{(-1)^{q+1-\beta} \sum_{p=0}^{\beta} \sum_{Lp \subset T\beta} (-1)^p A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)}\right\} \\
&\quad - \sum_{p=0}^{q+1} \sum_{Lp \subset R - I_{r-q-1}} (-1)^p A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)} \\
&= -\sum_{\beta=1}^{q+1} \sum_{T\beta \subset R - I_{r-q-1}} \left\{(-1)^{q+1-\beta} \sum_{p=0}^{\beta} \sum_{Lp \subset T\beta} (-1)^p A_{(i_1\cdots i_{r-q-1} l_1 \cdots l_p)}\right\} \\
&= -\left[\sum_{T_1 \subset R - I_{r-q-1}} (-1)^q \sum_{p=0}^{1} \sum_{Lp \subset T_1} C_{(p)} + \sum_{T_2 \subset R - I_{r-q-1}} (-1)^{q-1} \sum_{p=0}^{2} \sum_{Lp \supset T_2} C_{(p)}\right.
\end{aligned}
$$

$$+\cdots\cdots+\sum_{T_q\subset R-I_{r-q-1}}(-1)\sum_{p=0}^{q}\sum_{L_p\subset T_q}C_{(p)}+\sum_{T_{q+1}\subset R-I_{r-q-1}}(-1)^0\sum_{p=0}^{q+1}\sum_{L_p\subset T_{q+1}}C_{(p)}\Big]$$

except for the coefficient $\dfrac{1}{N_{r-q-1}}$.

The coefficient of $C_{(0)}$ in (4.35) is

$$(4.36)\quad -\Big[(-1)^q{}_{q+1}C_1+(-1)^{q-1}_{q+1}C_2+\cdots\cdots+(-1)_{q+1}C_q+{}_{q+1}C_{q+1}\Big]C_{(0)}$$
$$=(-1)^{q+1}A_{(i_1\cdots i_{r-q-1})}$$

and the partial sum of (4.35) for $1\leqq h\leqq q$ is given by

$$(4.37)\quad -\Big[\sum_{T_h\subset R-I_{r-q-1}}(-1)^{q-h+1}\sum_{L_h\subset T_h}C_{(h)}+\sum_{T_{h+1}\supset R-I_{r-q-1}}(-1)^{q-h}\sum_{L_h\subset T_h}C_{(h)}$$
$$+\cdots\cdots+\sum_{T_q\subset R-I_{r-q-1}}(-1)\sum_{L_h\subset T_q}C_{(h)}+\sum_{T_{q+1}\subset R-I_{r-q-1}}(-1)^0\sum_{L_h\subset T_{q+1}}C_{(h)}\Big]$$
$$=-\Big[(-1)^{q-h+1}_{q-h+1}C_0+(-1)^{q-h}_{q-h+1}C_1+(-1)^{q-h-1}_{q-h+1}C_2$$
$$+\cdots\cdots+{}_{q-h+1}C_{q-h+1}\Big]\sum_{L_h\subset R-I_{r-q-1}}C_{(h)}=0$$

and finally that of $C_{(q+1)}$ is

$$(4.38)\quad -\sum_{L_{q+1}\subset R-I_{r-q-1}}(-1)^{q+1}A_{(i_1\cdots i_{r-q-1}l_1\cdots l_{q+1})}=-(-1)^{q+1}A_{(12\cdots r)}$$

and hence (4.33) is composed of

$$(4.39)\quad \frac{(-1)^{q+1}}{B_{(12\cdots r)}}\Big[A_{(i_1\cdots i_{r-q-1})}-A_{(12\cdots r)}\Big]$$

and other remaining terms, among which the partial sum for $\alpha+\beta=c$ is given by

$$(4.40)\quad -\sum_{\beta=1}^{c}\sum_{T_\beta\subset R-I_{r-q-1}}\Big\{\sum_{\alpha=0}^{c-\beta}\sum_{S_\alpha\subset R-(I_{r-q-1}\cup T_\beta)}\frac{(-1)^\alpha}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_\beta s_1\cdots s_\alpha)}}$$
$$\cdot\sum_{p=0}^{\beta}\sum_{L_p\subset T_\beta}(-1)^pA_{(i_1\cdots i_{r-q-1}l_1\cdots l_p)}\Big\}.$$

This is divided into three parts, the sum for $p=0$, the sum for $p=h$ $(1\leqq h\leqq c-1)$ and the sum for $p=c$. These are evaluated in (4.43) (4.44) and (4.45) respectively, where some cumbersome considerations about the number of combinations are needed in simplifying the notation of summation, and the notations

$$(4.41)\quad D_{(t_1\cdots t_\beta s_1\cdots s_{c-\beta})}=\frac{A_{(i_1\cdots i_{r-q-1})}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_\beta s_1\cdots s_{c-\beta})}}$$

$$(4.42)\quad E_{(t_1\cdots t_hl_1\cdots l_hs_1\cdots s_{c-h})}=\frac{A_{(i_1\cdots i_{r-q-1}l_1\cdots l_h)}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_hs_1\cdots s_{c-h})}}.$$

are used.

$$(4.43)\quad -\sum_{\beta=1}^{c}\sum_{T\beta\subset R-I_{r-q-1}}\left\{\sum_{\alpha=0}^{c-\beta}\sum_{S\alpha\subset R-(I_{r-q-1}\cup T\beta)}\frac{(-1)^{\alpha}A_{(i_1\cdots i_{r-q-1})}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_\beta s_1\cdots s_\alpha)}}\right\}$$

$$=-\sum_{\beta=1}^{c}\sum_{T\beta\subset R-I_{r-q-1}}\sum_{S_{c-\beta}\subset R-(I_{r-q-1}\cup T\beta)}(-1)^{c-\beta}D_{(t_1\cdots t_\beta s_1\cdots s_{c-\beta})}$$

$$=-\sum_{T_c\subset R-I_{r-q-1}}(-1)^{0}D_{(t_1\cdots t_c)}-\sum_{T_{c-1}\subset R-I_{r-q-1}}\sum_{S_1\subset R-(I_{r-q-1}\cup T_{c-1})}(-1)^{1}D_{(t_1\cdots t_{c-1}s_1)}$$

$$-\cdots\cdots\cdots-\sum_{T_1\subset R-I_{r-q-1}}\sum_{S_{c-1}\subset R-(I_{r-q-1}\cup T_1)}(-1)^{c-1}D_{(t_1 s_1\cdots s_{c-1})}$$

$$=-\left[{}_cC_0(-1)^0+{}_cC_1(-1)^1+\cdots\cdots+{}_cC_{c-1}(-1)^{c-1}\right]\sum_{T_c\subset R-I_{r-q-1}}D_{(t_1\cdots t_c)}$$

$$=(-1)^{c}\sum_{T_c\subset R-I_{r-q-1}}D_{(t_1\cdots t_c)}$$

$$=(-1)^{c}\sum_{T_c\subset R-I_{r-q-1}}\frac{A_{(i_1\cdots i_{r-q-1})}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\ .$$

$$(4.44)\quad -\sum_{T_h\subset R-I_{r-q-1}}\sum_{S_{c-h}\subset R-(I_{r-q-1}\cup T_h)}\sum_{L_h\subset T_h}(-1)^{c}E_{(t_1\cdots t_h l_1\cdots l_h s_1\cdots s_{c-h})}$$

$$-\sum_{T_{h+1}\subset R-I_{r-q-1}}\sum_{S_{c-h}\subset R-(I_{r-q-1}\cup T_{h+1})}\sum_{L_h\subset T_{h+1}}(-1)^{c-1}E_{(t_1\cdots t_{h+1}l_1\cdots l_h s_1\cdots s_{c-h-1})}$$

$$-\sum_{T_{h+2}\subset R-I_{r-q-1}}\sum_{S_{c-h}\subset R-(I_{r-q-1}\subset T_{h+2})}\sum_{L_h\subset T_{h+2}}(-1)^{c-2}E_{(t_1\cdots t_{h+2}l_1\cdots l_h s_1\cdots s_{c-h-2})}$$

$$-\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\sum_{T_c\subset R-I_{r-q-1}}\sum_{L_h\subset T_c}(-1)^{h}E_{(t_1\cdots t_c l_1\cdots l_h)}$$

$$=-\left[\sum_{j=0}^{h}(-1)^{c-j}{}_{c-h}C_j\right]\sum_{T_h\subset R-I_{r-q-1}}\sum_{S_{c-h}\subset R-(I_{r-q-1}\cup T_h)}\sum_{L_h\subset T_h}E_{(t_1\cdots t_h l_1\cdots l_h s_1\cdots s_{c-h})}$$

$$=0.$$

$$(4.45)\quad -\sum_{T_c\subset R-I_{r-q-1}}\frac{(-1)^{c}A_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\ .$$

Now (4.33) is simplified to

$$(4.46)\quad X_{i_1\cdots i_{r-q-1}}B_{(i_1\cdots i_{r-q-1})}=\frac{1}{N_{r-q-1}}\left[\frac{(-1)^{q-1}}{B_{(12\cdots r)}}\left\{A_{(i_1\cdots i_{r-q-1})}-A_{(12\cdots r)}\right\}\right.$$
$$\left.+\sum_{c=1}^{q}\sum_{T_c\subset R-I_{r-q-1}}(-1)^{c}\left\{\frac{A_{(i_1\cdots i_{r-q-1})}-A_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\right\}\right]$$

and we have

$$(4.47)\quad X_{i_1\cdots i_{r-q-1}}=\frac{1}{N_{r-q-1}}\left[(-1)^{q+1}\left\{\frac{1}{B_{(12\cdots r)}}-\frac{1}{B_{(i_1\cdots i_{r-q-1})}}\right\}\right.$$
$$\left.+\sum_{c=1}^{q}\sum_{T_c\subset R-I_{r-q-1}}(-1)^{c}\left\{\frac{1}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}-\frac{1}{B_{(i_1\cdots i_{r-q-1})}}\right\}\right]$$
$$=\frac{1}{N_{r-q-1}}\left[\frac{(-1)^{q+1}}{B_{(12\cdots r)}}-\frac{(-1)^{q+1}}{B_{(i_1\cdots i_{r-q-1})}}+\sum_{c=1}^{q}\sum_{T_c\subset R-I_{r-q-1}}(-1)^{c}\frac{1}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\right.$$
$$\left.-\sum_{c=1}^{q}\sum_{T_c\subset R-I_{r-q-1}}(-1)^{c}\frac{1}{B_{(i_1\cdots i_{r-q-1})}}\right]$$

$$= \frac{1}{N_{r-q-1}}\left[\frac{(-1)^{q+1}}{B_{(12\cdots r)}} + \frac{1}{B_{(i_1\cdots i_{r-q-1})}} + \sum_{c=1}^{q}\sum_{T_c\subset R-I_{r-q-1}} \frac{(-1)^c}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\right]$$
$$= \frac{1}{N_{r-q-1}}\left[\frac{(-1)^{q+1}}{B_{(12\cdots r)}} + \sum_{c=0}^{q}\sum_{T_c\subset R-I_{r-q-1}} \frac{(-1)^c}{B_{(i_1\cdots i_{r-q-1}t_1\cdots t_c)}}\right].$$

Thus we have proved that the solution of (4.29) is given by (4.22).

Finally (4.23) is obtained by inserting (4.20), (4.21) and (4.22) in (4.30) in the following way.

After inserting them in (4.30), we have

$$(4.48)\quad X_E B = -\sum_{k=1}^{r-1}\sum_{I_k\subset R}\left[\frac{1}{\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}}\left\{\frac{(-1)^{r-k}}{B_{(12\cdots r)}} + \sum_{\alpha=0}^{r-k-1}\sum_{S_\alpha\subset R-I_k}\frac{(-1)^\alpha}{B_{(i_1\cdots i_k s_1\cdots s_\alpha)}}\right\}\right.$$
$$\left.\cdot\frac{1}{n_{i_1}\cdots n_{i_k}}\sum_{p=0}^{k}\sum_{L_p\subset I_k}(-1)^p A_{(l_1\cdots l_p)}\right]$$
$$= -\frac{1}{\prod_{j=0}^{r} n_j}\left[\sum_{k=1}^{r-1}\sum_{I_k\subset R}\frac{(-1)^{r-k}}{B_{(12\cdots r)}}\sum_{p=0}^{k}\sum_{L_p\subset I_k}(-1)^p A_{(l_1\cdots l_p)}\right]$$
$$-\frac{1}{\prod_{j=0}^{r} n_j}\left[\sum_{k=1}^{r-1}\sum_{I_k\subset R}\sum_{\alpha=0}^{r-k-1}\sum_{S_\alpha\subset R-I_k}\frac{(-1)^\alpha}{B_{(i_1\cdots i_k s_1\cdots s_\alpha)}}\sum_{p=0}^{k}\sum_{L_p\subset I_k}(-1)^p A_{(l_1\cdots l_p)}\right].$$

By writing $G_{(p)}=A_{(l_1\cdots l_p)}/B_{(12\cdots r)}$ the first term is equal to

$$(4.49)\quad -\sum_{k=1}^{r-1}\sum_{I_k\subset R}\sum_{p=0}^{k}\sum_{L_p\subset I_k}\frac{(-1)^{r-k+p}A_{(l_1\cdots l_p)}}{B_{(12\cdots r)}}$$
$$= -\sum_{I_1\subset R}\sum_{p=0}^{1}\sum_{L_p\subset I_1}(-1)^{r-1+p}G_{(p)} - \sum_{I_2\subset R}\sum_{p=0}^{2}\sum_{L_p\subset I_2}(-1)^{r-2+p}G_{(p)}$$
$$-\cdots\cdots-\sum_{I_{r-1}\subset R}\sum_{p=0}^{r-1}\sum_{L_p\subset I_{r-1}}(-1)^{r-(r-1)+p}G_{(p)}.$$

The sum for $p=0$ and the sum for $p=h$ $(1\leq h\leq r-1)$ are given by (4.50) and (4.51) respectively.

$$(4.50)\quad -\sum_{j=1}^{r-1}{}_rC_j(-1)^{r-j}G_{(0)} = -\left[\sum_{j=0}^{r}{}_rC_{r-j}(-1)^{r-j} - {}_rC_r(-1)^r - {}_rC_0\right]G_{(0)}$$
$$= -\left[-(-1)^r - 1\right]G_{(0)} = \left\{(-1)^r+1\right\}\frac{A}{B_{(12\cdots r)}}.$$
$$(4.51)\quad -\sum_{I_h\subset R}\sum_{L_h\subset I_h}(-1)^r G_{(h)} - \sum_{I_{h+1}\subset R}\sum_{L_h\subset I_{r+1}}(-1)^{r-1}G_{(h)}$$
$$-\cdots\cdots-\sum_{I_{r-1}\subset R}\sum_{L_h\subset I_{r-1}}(-1)^{h+1}G^{(h)}$$
$$= -\left[\sum_{j=0}^{r-h-1}(-1)^{r-j}{}_{r-h}C_j\right]\sum_{I_h\subset R}\sum_{L_h\subset I_h}G_{(h)} = (-1)^h\sum_{I_h\subset R}\frac{A_{(i_1\cdots i_h)}}{B_{(12\cdots r)}}.$$

On the other hand the second term in (4.48) is equal to

$$(4.52)\quad -\sum_{c=1}^{r-1}\sum_{k=1}^{c}\sum_{I_k\subset R}\sum_{S_{c-k}\subset R-I_k}\frac{(-1)^{c-k}}{B_{(i_1\cdots i_k s_1\cdots s_{c-k})}}\sum_{p=0}^{k}\sum_{L_p\subset I_k}(-1)^p A_{(l_1\cdots l_p)}$$

The sum for $p=0$, $p=h$ $(1\leq h\leq c-1)$ and $p=c$ are given by

$$(4.53)\quad -\sum_{I_1\subset R}\ \sum_{S_{c-1}\subset R-I_1}\frac{(-1)^{c-1}A}{B_{(i_1s_1\cdots s_{c-1})}}-\sum_{I_2\subset R}\ \sum_{S_{c-2}\subset R-I_2}\frac{(-1)^{c-2}A}{B_{(i_1i_2s_1\cdots s_{c-2})}}$$

$$-\cdots\cdots-\sum_{I_c\subset R}\frac{(-1)^0A}{B_{(i_1\cdots i_c)}}$$

$$=(-1)^c\sum_{I_c\subset R}\frac{A}{B_{(i_1\cdots i_c)}}.$$

$$(4.54)\quad -\sum_{I_h\subset R}\ \sum_{S_{c-h}\subset R-I_h}\frac{(-1)^{c-h}}{B_{(i_1\cdots i_hs_1\cdots s_{c-h})}}\sum_{L_h\subset I_h}A_{(l_1\cdots l_h)}$$

$$-\sum_{I_{h+1}\subset R}\ \sum_{S_{c-h-1}\subset R-I_{h+1}}\frac{(-1)^{c-h-1}}{B_{(i_1\cdots i_{h+1}s_1\cdots s_{c-h-1})}}\sum_{L_h\subset I_{h+1}}A_{(l_1\cdots l_h)}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\sum_{I_c\subset R}\frac{(-1)^0}{B_{(i_1\cdots i_c)}}\sum_{L_h\subset I_c}A_{(l_1\cdots l_h)}$$

$$=-\left[\sum_{j=0}^{c-h}(-1)^{c-h-j}{}_{c-h}C_j\right]\sum_{L_h\subset R}\ \sum_{S_{c-h}\subset R-L_h}\frac{(-1)^{c-h}A_{(l_1\cdots l_h)}}{B_{(l_1\cdots l_hs_1\cdots s_{c-h})}}$$

$$=0.$$

$$(4.55)\quad -\left[\sum_{I_c\subset R}\frac{(-1)^0}{B_{(i_1\cdots i_c)}}\sum_{L_c\subset I_c}(-1)^cA_{(l_1\cdots l_c)}\right]$$

$$=-\sum_{I_c\subset R}\frac{(-1)^cA_{(i_1\cdots i_c)}}{B_{(i_1\cdots i_c)}}.$$

The combination of (4.48), $\cdots\cdots$, (4.55) yields

$$(4.56)\quad X_EB=-\frac{1}{\prod_{j=0}^{r}n_j}\left[\frac{\{(-1)^r+1\}A}{B_{(12\cdots r)}}+\sum_{h=1}^{r-1}\sum_{I_h\subset R}\frac{(-1)^hA_{(i_1\cdots i_h)}}{B_{(12\cdots r)}}\right.$$

$$\left.+\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^cA}{B_{(i_1\cdots i_c)}}-\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^cA_{(i_1\cdots i_c)}}{B_{(i_1\cdots i_c)}}\right].$$

On the other hand $A^{(12\cdots r)}$ is the sum over the null index set and is equal to 0, and Lemma 4.2 should holds true even if $T_\alpha=(t_1,\cdots\cdots,t_\alpha)$ is the null set, and we have

$$(4.57)\quad A^{(12\cdots r)}=\frac{1}{\prod_{j=0}^{r}n_j}\sum_{h=0}^{r}\sum_{I_h\subset R}(-1)^hA_{(i_1\cdots i_h)}$$

$$=\frac{1}{\prod_{j=0}^{r}n_j}\left[A+\sum_{h=1}^{r-1}\sum_{I_h\subset R}(-1)^hA_{(i_1\cdots i_h)}+(-1)^rA_{(12\cdots r)}\right]=0$$

This is equivalent to

$$(4.58)\quad \sum_{h=1}^{r-1}\sum_{I_h\subset R}(-1)^hA_{(i_1\cdots i_h)}=-\{A+(-1)^rA_{(12\cdots r)}\}$$

Inserting (4.58) in (4.56) we have finally

$$(4.59)\quad X_k = \frac{1}{\prod_{j=0}^{r} n_j B}\Bigg[\frac{\{(-1)^r+1\}A}{B_{(12\cdots r)}} - \frac{A+(-1)^r A_{(12\cdots r)}}{B_{(12\cdots r)}}$$

$$+\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c A}{B_{(i_1\cdots i_c)}} - \sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c A_{(i_1\cdots i_c)}}{B_{(i_1\cdots i_c)}}\Bigg]$$

$$= \frac{1}{\prod_{j=0}^{r} n_j B}\Bigg[\frac{(-1)^r A}{B_{(12\cdots r)}} - \frac{(-1)^r A_{(12\cdots r)}}{B_{(12\cdots r)}}$$

$$+\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c A}{B_{(i_1\cdots i_c)}} - \sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c A_{(i_1\cdots i_c)}}{B_{(i_1\cdots i_c)}}$$

$$= \frac{1}{\prod_{j=0}^{r} n_j}\Bigg[\frac{(-1)^r}{B_{(12\cdots r)}} - \frac{(-1)^r}{B} + \sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c}{B_{(i_1\cdots i_c)}} - \sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c}{B}\Bigg]$$

$$= \frac{1}{\prod_{j=0}^{r} n_j}\Bigg[\frac{(-1)^r}{B_{(12\cdots r)}} + \frac{1}{B} + \sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c}{B_{(i_1\cdots i_c)}}\Bigg].$$

Thus we have completed the proof of this lemma and also Theorem 4.2.

4.3 *The joint density function.* We have derived the determinant and the inverse of the variance matrix, and what we have to do is to derive the joint density function as the generalization of (3.17), which is ennunciated in

THEOREM 4.3. The joint density function of $x_{t_0 t_1\cdots t_r}$ is given by

$$(4.60)\quad f(\mathbf{X}) = (2\pi)^{-n_0 n_1\cdots n_r/2} B^{-1/2}\prod_{k=1}^{r}\prod_{I_k\subset R}\{B_{(i_1\cdots i_k)}\}^{-(n_{i1}-1)(n_{i2}-1)\cdots(n_{ik}-1)/2}\sigma_0^{-(n_0-1)n_1\cdots n_r/2}$$

$$\cdot exp\Bigg[-\frac{1}{2}\Bigg\{\prod_{j=0}^{r} n_j(\bar{X}-\mu)^2\frac{1}{B} + \sum_{k=1}^{r}\sum_{I_k\subset R}\frac{S_{(i_1\cdots i_k)}}{B_{(i_1\cdots i_k)}} + \frac{S_0}{\sigma_0}\Bigg\}\Bigg],$$

where

$$(4.61)\quad \bar{X}_{t_{l_1}\cdots t_{l_\beta}} = \frac{1}{\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}}\sum_{\substack{t_{j_1},\cdots,t_{j_{r-\beta}}\\ J_{r-\beta}\subset R-L_\beta}}\sum_{t_0} x_{t_0 t_1\cdots t_r} \qquad (L_\beta\subset R,\ \beta=1,\cdots\cdots,r),$$

$$(4.62)\quad \bar{X} = \frac{1}{\prod_{j=0}^{r} n_j}\sum_{t_0,t_1,\cdots,t_r} x_{t_0 t_1\cdots t_r},$$

$$(4.63)\quad S_{(i_1\cdots i_k)} = \prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}\sum_{t_{i_1},\cdots,t_{i_k}}\Bigg\{\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\bar{X}_{t_{l_1}\cdots t_{l_\beta}}\Bigg\}^2,$$

$$(4.64)\quad S_0 = \sum_{t_0,t_1,\cdots,t_r}(x_{t_0 t_1\cdots t_r} - \bar{X}_{t_1\cdots t_r})^2.$$

PROOF. In this proof we shall use the convention that if $(t_{l_1},\cdots\cdots,t_{l_\beta})$ is the null set $\bar{X}_{t_{l_1}\cdots t_{l_\beta}} = \bar{X}$. As the density function should be the multivariate normal density function, the constant factor in (4.60) is easily derived from Theorem 4.2, and there remains only to derive the quadratic form of $x_{t_0 t_1\cdots t_r}$.

Now let us introduce new variables defined by

$$(4.65)\quad u_{t_0t_1\cdots t_r}=x_{t_0t_1\cdots t_r}-\mu,$$

$$(4.66)\quad U_{t_{i_1}\cdots t_{i_k}}=\sum_{\substack{t_{j_1},\cdots,t_{j_{r-k}}\\ J_{r-k}\subset R-I_k}}\sum_{t_0}u_{t_0t_1\cdots t_r},$$

$$(4.67)\quad U=\sum_{t_0,t_1,\cdots,t_r}u_{t_0t_1\cdots t_r}=\prod_{j=0}^{r}n_j\bar{U},$$

$$(4.68)\quad \bar{U}_{t_{i_1}\cdots t_{i_k}}=\frac{1}{\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}}U_{t_{i_1}\cdots t_{i_k}},$$

and we shall use the convention for $\bar{U}_{t_{l_1}\cdots t_{l_\beta}}$ same to that we have made for $\bar{X}_{t_{l_1}\cdots t_{l_\beta}}$ at the beginning of the proof.

As the inverse matrix has already been derived in Theorem 4.2., the density should be written as $exp\,[-S/2]$ except for the constant factor, where

$$\begin{aligned}(4.69)\quad S=&X_E\left(\sum_{t_0,t_1,\cdots,t_r}u_{t_0t_1\cdots t_r}\right)^2+\sum_{k=1}^{r-1}\sum_{I_k\subset R}X_{i_1\cdots i_k}\left\{\sum_{t_{i_1},\cdots,t_{i_k}}\left(\sum_{\substack{t_{j_1},\cdots,t_{j_{r-k}}\\ J_{r-k}\subset R-I_k}}\sum_{t_0}u_{t_0t_1\cdots t_r}\right)^2\right\}\\
&+X_{12\cdots r}\sum_{t_1,\cdots,t_r}\left(\sum_{t_0}u_{t_0t_1\cdots t_r}\right)^2+X_0\sum_{t_0,t_1,\cdots,t_r}u^2_{t_0t_1\cdots t_r}\\
=&X_EU^2+\sum_{k=1}^{r-1}\sum_{I_k\subset R}\left\{X_{i_1\cdots i_k}\sum_{t_{i_1},\cdots,t_{i_k}}U^2_{t_{i_1}\cdots t_{i_k}}\right\}\\
&+X_{12\cdots r}\sum_{t_1,\cdots,t_r}U^2_{t_1\cdots t_r}+X_0\sum_{t_0,t_1,\cdots,t_r}u^2_{t_0t_1\cdots t_r}\\
=&U^2\frac{1}{\prod_{j=0}^{r}n_j}\left[\frac{(-1)^r}{B_{(12\cdots r)}}+\frac{1}{B}+\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{(-1)^c}{B_{(i_1\cdots i_c)}}\right]\\
&+\sum_{k=1}^{r-1}\sum_{I_k\subset R}\left(\sum_{t_{i_1},\cdots,t_{i_k}}U^2_{t_{i_1}\cdots t_{i_k}}\right)\frac{1}{\prod_{j=0}^{r}n_j^{1-\delta^j_{i_1\cdots i_k}}}\left[\frac{(-1)^{r-k}}{B_{(12\cdots r)}}+\sum_{\alpha=0}^{r-k-1}\sum_{S_\alpha\subset R-I_k}\frac{(-1)^\alpha}{B_{(i_1\cdots i_ks_1\cdots s_\alpha)}}\right]\\
&+\sum_{t_1,\cdots,t_r}U^2_{t_1\cdots t_r}\frac{1}{n_0}\left[\frac{1}{B_{(12\cdots r)}}-\frac{1}{\sigma_0}\right]+\frac{1}{\sigma_0}\sum_{t_0,t_1,\cdots,t_r}u^2_{t_0t_1\cdots t_r}.\end{aligned}$$

After evaluating the coefficient to $\frac{1}{B_{(i_1\cdots i_c)}}$ and $\frac{1}{B_{(12\cdots r)}}$ in (4.69) which are given by

$$\begin{aligned}(4.70)\quad D_{(i_1\cdots i_c)}=&U^2\frac{1}{\prod_{j=0}^{r}n_j}(-1)^c\\
&+\sum_{\beta=1}^{c}\sum_{L_\beta\subset I_c}\left(\sum_{t_{l_1},\cdots,t_{l_\beta}}U^2_{t_{l_1}\cdots t_{l_\beta}}\right)\frac{1}{\prod_{j=0}^{r}n_j^{1-\delta^j_{l_1\cdots l_\beta}}}(-1)^{c-\beta}\\
=&\sum_{\beta=0}^{c}\sum_{L_\beta\subset I_c}\frac{(-1)^{c-\beta}}{\prod_{j=0}^{r}n_j^{1-\delta^j_{l_1\cdots l_\beta}}}\left(\sum_{t_{l_1},\cdots,t_{l_\beta}}U^2_{t_{l_1}\cdots t_{l_\beta}}\right)\end{aligned}$$

and

$$(4.71)\quad D_{(12\cdots r)}=U^2\frac{1}{\prod_{j=0}^{r} n_j}(-1)^r$$
$$+\sum_{k=1}^{r-1}\sum_{I_k\subset R}\Big(\sum_{t_{i1},\cdots,t_{ik}}U^2_{t_{i1}\cdots t_{ik}}\Big)\frac{(-1)^{r-k}}{\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}}+\sum_{t_1,\cdots,t_r}U^2_{t_1\cdots t_r}\frac{1}{n_0}$$
$$=\sum_{k=0}^{r}\sum_{I_k\subset R}\frac{(-1)^{r-k}}{\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}}\Big(\sum_{t_{i1},\cdots,t_{ik}}U^2_{t_{i1}\cdots t_{ik}}\Big)$$

respectively, we have (4.69) is equal to

$$(4.72)\quad S=\sum_{c=1}^{r-1}\sum_{I_c\subset R}\frac{D_{(i_1\cdots i_c)}}{B_{(i_1\cdots i_c)}}+\frac{D_{(12\cdots r)}}{B_{(12\cdots r)}}+U^2\frac{1}{\prod_{j=0}^{r} n_j}\cdot\frac{1}{B}$$
$$+\Big[\sum_{t_0,t_1,\cdots,t_r}u^2_{t_0t_1\cdots t_r}-\frac{1}{n_0}\sum_{t_1,\cdots,t_r}U^2_{t_1\cdots t_r}\Big]\frac{1}{\sigma_0}$$
$$=\sum_{k=1}^{r}\sum_{I_k\subset R}\frac{D_{(i_1\cdots i_k)}}{B_{(i_1\cdots i_k)}}+U^2\frac{1}{\prod_{j=0}^{r} n_j}\frac{1}{B}$$
$$+\Big[\sum_{t_0,t_1,\cdots,t_r}u^2_{t_0t_1\cdots t_r}-\frac{1}{n_0}\sum_{t_1,\cdots,t_r}U^2_{t_1\cdots t_r}\Big]\frac{1}{\sigma_0}$$
$$=\sum_{k=1}^{r}\sum_{I_k\subset R}\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}\frac{(-1)^{k-\beta}}{\prod_{j=0}^{r} n_j^{1-\delta^j_{l_1\cdots l_\beta}}}\Big(\sum_{t_{l1},\cdots,t_{l\beta}}U^2_{t_{l1}\cdots t_{l\beta}}\Big)\frac{1}{B_{(i_1\cdots i_k)}}$$
$$+U^2\frac{1}{\prod_{j=0}^{r} n_j}\frac{1}{B}+\Big[\sum_{t_0,t_1,\cdots,t_r}u^2_{t_0t_1\cdots t_r}-\frac{1}{n_0}\sum_{t_1,\cdots,t_r}U^2_{t_1\cdots t_r}\Big]\frac{1}{\sigma_0}$$
$$=\sum_{k=1}^{r}\sum_{I_k\subset R}\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\prod_{j=0}^{r} n_j^{1-\delta^j_{l_1\cdots l_\beta}}\Big(\sum_{t_{l1},\cdots,t_{l\beta}}\bar U^2_{t_{l1}\cdots t_{l\beta}}\Big)\frac{1}{B_{(i_1\cdots i_k)}}$$
$$+\sum_{j=0}^{r}n_j\bar U^2\frac{1}{B}+\Big[\sum_{t_0,t_1,\cdots,t_r}u^2_{t_0t_1\cdots t_r}-n_0\sum_{t_1,\cdots,t_r}\bar U^2_{t_1\cdots t_r}\Big]\frac{1}{\sigma_0}$$
$$=\sum_{k=1}^{r}\sum_{I_k\subset R}\prod_{j=0}^{r} n_j^{1-\delta^j_{i_1\cdots i_k}}\Big[\sum_{t_{i1},\cdots,t_{ik}}\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\bar U^2_{t_{l1}\cdots t_{l\beta}}\Big]\frac{1}{B_{(i_1\cdots i_k)}}$$
$$+\prod_{j=0}^{r}n_j\bar U^2\frac{1}{B}+\Big[\sum_{t_0,t_1,\cdots t_r}u^2_{t_0t_1\cdots t_r}-n_0\sum_{t_1,\cdots,t_r}\bar U^2_{t_1\cdots t_r}\Big]\frac{1}{\sigma_0}\,.$$

Here we need to prove the following

LEMMA 4.4.

$$(4.73)\quad \sum_{t_{i1},\cdots,t_{ik}}\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\bar U^2_{t_{l1}\cdots t_{l\beta}}=\sum_{t_{i1},\cdots,t_{ik}}\Big[\sum_{\beta=0}^{k}\sum_{L_\beta\subset I_k}(-1)^{k-\beta}\bar U_{t_{l1}\cdots t_{l\beta}}\Big]^2$$

PROOF. Proof is given by making use of the mathematical induction. In case $k=1$, the proof is given by

$$(4.74)\quad \sum_{t_{i_1}}\{(-1)\,\bar{U}^2+(-1)^0\bar{U}^2_{t_{i_1}}\}=\sum_{t_{i_1}}(\bar{U}^2_{t_{i_1}}-\bar{U}^2)$$
$$=\sum_{t_{i_1}}\bar{U}^2_{t_{i_1}}-2\sum_{t_{i_1}}\bar{U}^2+\sum_{t_{i_1}}\bar{U}^2$$
$$=\sum_{t_{i_1}}\bar{U}^2_{t_{i_1}}-2\sum_{t_{i_1}}\bar{U}_{t_{i_1}}\bar{U}+\sum_{t_{i_1}}\bar{U}^2$$
$$=\sum_{t_{i_1}}(\bar{U}_{t_{i_1}}-\bar{U})^2.$$

Further assuming (4.73) to be valid in case $k=h$ we have

$$(4.75)\quad \sum_{t_{i_1},\cdots,t_{i_h},t_{i_{h+1}}}\ \sum_{\beta=0}^{h+1}\ \sum_{L_\beta\subset I_{h+1}}(-1)^{h+1-\beta}\bar{U}^2_{t_{l_1}\cdots t_{l_\beta}}$$
$$=\sum_{t_{i_1}\cdots t_{i_{h+1}}}\ \sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}^2_{t_{l_1}\cdots t_{l_\beta}t_{i_{h+1}}}-\sum_{t_{i_1},\cdots,t_{i_{h+1}}}\ \sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}^2_{t_{l_1}\cdots t_{l_\beta}}$$
$$=\sum_{t_{i_{h+1}}}\Big[\sum_{t_{i_1},\cdots,t_{i_h}}\ \sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}^2_{t_{l_1}\cdots t_{l_\beta}t_{i_{h+1}}}-\sum_{t_{i_1},\cdots,t_{i_h}}\ \sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}^2_{t_{l_1}\cdots t_{l_\beta}}\Big]$$
$$=\sum_{t_{i_{h+1}}}\Big[\sum_{t_{i_1},\cdots,t_{i_h}}\Big\{\sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}t_{i_{h+1}}}\Big\}^2-\sum_{t_{i_1},\cdots,t_{i_h}}\Big\{\sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}}\Big\}^2\Big]$$
$$=\sum_{t_{i_1},\cdots,t_{i_{h+1}}}\Big[\Big\{\sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}t_{i_{h+1}}}\Big\}^2-\Big\{\sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}}\Big\}^2\Big].$$

And by writing

$$(4.76)\quad \bar{Z}_{t_{i_1}\cdots t_{i_h}}=\sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}},$$

$$(4.77)\quad \bar{Z}_{t_{i_1}\cdots t_{i_h}(t_{i_{h+1}})}=\sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}t_{i_{h+1}}},$$

(4.75) is equal to

$$(4.78)\quad \sum_{t_{i_1},\cdots,t_{i_{h+1}}}\Big[\bar{Z}^2_{t_{i_1}\cdots t_{i_h}(t_{i_{h+1}})}-\bar{Z}^2_{t_{i_1}\cdots t_{i_h}}\Big]$$
$$=\sum_{t_{i_1},\cdots,t_{i_{h+1}}}\Big[\bar{Z}^2_{t_{i_1}\cdots t_{i_h}(t_{i_{h+1}})}-2\bar{Z}^2_{t_{i_1}\cdots t_{i_h}}+\bar{Z}^2_{t_{i_1}\cdots t_{i_h}}\Big]$$
$$=\sum_{t_{i_1},\cdots,t_{i_{h+1}}}\Big[\bar{Z}^2_{t_{i_1}\cdots t_{i_h}(t_{i_{h+1}})}-2\bar{Z}_{t_{i_1}\cdots t_{i_h}}\bar{Z}_{t_{i_1}\cdots t_{i_h}(t_{i_{h+1}})}+\bar{Z}^2_{t_{i_1}\cdots t_{i_h}}\Big]$$
$$=\sum_{t_{i_1},\cdots,t_{i_{h+1}}}\Big[\bar{Z}_{t_{i_1}\cdots t_{i_h}(t_{i_{h+1}})}-\bar{Z}_{t_{i_1}\cdots t_{i_h}}\Big]^2$$
$$=\sum_{t_{i_1},\cdots,t_{i_{h+1}}}\Big[\sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}t_{i_{h+1}}}-\sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}}\Big]^2$$
$$=\sum_{t_{i_1},\cdots,t_{i_{h+1}}}\Big[\sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}t_{i_{h+1}}}+\sum_{\beta=0}^{h}\ \sum_{L_\beta\subset I_h}(-1)^{h+1-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}}\Big]^2$$
$$=\sum_{t_{i_1},\cdots,t_{i_{h+1}}}\Big[\sum_{\beta=1}^{h+1}\ \sum_{\substack{L_\beta\subset I_{h+1}\\ L_\beta\ni i_{h+1}}}(-1)^{h-\beta+1}\bar{U}_{t_{l_1}\cdots t_{l_\beta}}+\sum_{\beta=0}^{h}\ \sum_{\substack{L_\beta\subset I_{h+1}\\ L_\beta\not\ni i_{h+1}}}(-1)^{h+1-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}}\Big]^2$$
$$=\sum_{t_{i_1},\cdots,t_{i_{h+1}}}\Big[\sum_{\beta=0}^{h+1}\ \sum_{L_\beta\subset I_{h+1}}(-1)^{h+1-\beta}\bar{U}_{t_{l_1}\cdots t_{l_\beta}}\Big]^2$$

which completes the proof of the lemma.

By making use of this Lemma the first term in (4.72) comes to be equal to (4.79), and the second term is equal to (4.80), and finally the third is equal to (4.81);

$$(4.79)\quad \sum_{t_{i1},\cdots,t_{ik}}\left[\sum_{\beta=0}^{k}\sum_{L\beta\subset Ik}(-1)^{k-\beta}\bar{U}_{t_{l1}\cdots t_{l\beta}}\right]^2=\sum_{t_{i1},\cdots,t_{ik}}\left[\sum_{\beta=0}^{k}\sum_{L\beta\subset Ik}(-1)^{k-\beta}(\bar{X}_{t_{l1}\cdots t_{l\beta}}-\mu)\right]^2$$

$$=\sum_{t_{i1},\cdots,t_{ik}}\left[\sum_{\beta=0}^{k}\sum_{L\beta\subset Ik}(-1)^{k-\beta}\bar{X}_{t_{l1}\cdots t_{l\beta}}\right]^2.$$

$$(4.80)\quad \prod_{j=0}^{r} n_j(\bar{X}-\mu)^2\frac{1}{B}.$$

$$(4.81)\quad \left[\sum_{t_0,t_1,\cdots,t_r}(u_{t_0t_1\cdots t_r}-\bar{U}_{t_1\cdots t_r})^2\right]\frac{1}{\sigma_0}$$

$$=\sum_{t_0,t_1,\cdots,t_r}(x_{t_0t_1\cdots t_r}-\bar{X}_{t_1\cdots t_r})^2\frac{1}{\sigma_0}.$$

The combination of (4.79), (4.80) and (4.81) leads us to the completion of the proof.

4.4 *Estimation.* Finally we shall treat the problem of estimation of the variance components. By the usual estimates of the variance components we mean the usual ones, which is calculated as a linear form of a suitable number of mean squares in the table of the analysis of variance and is widely used as the estimates in the ordinary practice of statistical analysis. As we have already stated in the case of the 2-way layout, the completeness of the family of distribution of the sufficient statistics in our concern is yet in question, and we shall here make use of the notion of the minimum variance estimate due to Bhattacharyya to justify the usual estimates. Thus we have,

THEOREM 4.4. *In the r-way layout of random effect model, the minimum variance estimates of the variance components* $\sigma_{i_1\cdots i_k}$ $(I_k\subset R,\ k=1,\cdots\cdots,r)$ *are given by such linear forms of* $S_{(i_1\cdots i_k)}$ *and* S_0 *in (4.63) and (4.64) that these are unbiased, namely the usual estimates of the variance components, and that of the general mean is given by the sample total mean.*

PROOF. After taking the logarithm of the density function (4.60)

$$(4.82)\quad \ln f=K-\frac{1}{2}\ln B-\frac{1}{2}\sum_{k=1}^{r}\sum_{I_k\subset R}(n_{i_1}-1)(n_{i_2}-1)\cdots\cdots(n_{i_k}-1)\ln B_{(i_1\cdots i_k)}$$

$$-\frac{1}{2}(n_0-1)\,n_1n_2\cdots\cdots n_r\ln\sigma_0$$

$$-\frac{1}{2}\left[\prod_{j=0}^{r}n_j(\bar{X}-\mu)^2\frac{1}{B}+\sum_{k=1}^{r}\sum_{I_k\subset R}\frac{S_{(i_1\cdots i_k)}}{B_{(i_1\cdots i_k)}}+\frac{S_0}{\sigma_0}\right],$$

we have easily

$$(4.83)\quad B\prod_{j=0}^{r}n_j\frac{\partial f}{\partial\mu}=(\bar{X}-\mu)f,$$

$$(4.84)\quad \frac{2\sigma_0^2}{(n_0-1)n_1\cdots n_r}\frac{\partial f}{\partial\sigma_0}=\left\{\frac{S_0}{(n_0-1)n_1\cdots n_r}-\sigma_0\right\}f,$$

(4.85) $$\frac{2\{B_{(i_1\cdots i_k)}\}^2}{\prod\limits_{c=1}^{k}(n_{i_c}-1)}\frac{\partial f}{\partial B_{(i_1\cdots i_k)}}+\frac{(-1)^k\{B_{(i_1\cdots i_k)}\}^2}{\prod\limits_{j=0}^{r}n_j\cdot\prod\limits_{c=1}^{k}(n_{i_c}-1)}\frac{\partial^2 f}{\partial\mu^2}=\left[\frac{S_{(i_1\cdots i_k)}}{\prod\limits_{c=1}^{k}(n_{i_c}-1)}-B_{(i_1\cdots i_k)}\right]f,$$

$$(I_k\subset R,\ k=1,\cdots\cdots,r).$$

In view of the result given in Chapter II of Bhattacharyya [1] we observe that each one of these three relations shows the minimum variance estimates of the general mean μ, the variance of the error term σ_0 and $B_{(i_1\cdots i_k)}$ are given by the total mean $\bar{X}$, the mean square due to error $\dfrac{S_0}{(n_0-1)n_1n_2\cdots n_r}$, and $\dfrac{S_{(i_1i_2\cdots i_k)}}{(n_{i_1}-1)(n_{i_2}-1)\cdots(n_{i_k}-1)}$ respectively. The proof that the usual estimates of the variance components is of minimum variance can be obtained by taking a linear combination of a suitable number of equations in (4.85). For instance that the minimum variance estimate of $\sigma_{23\cdots r}$ is given by

(4.86) $$\frac{1}{n_0n_1}\left[\frac{S_{(23\cdots r)}}{(n_2-1)(n_3-1)\cdots(n_r-1)}-\frac{S_{(12\cdots r)}}{(n_1-1)(n_2-1)\cdots(n_r-1)}\right]$$

can be proved by noting

(4.87) $$\sigma_{23\cdots r}=\frac{B_{(23\cdots r)}-B_{(12\cdots r)}}{n_0n_1}$$

and by taking the linear form of (4.85) involving $B_{(23\cdots r)}$ and $B_{(12\cdots r)}$ and by using the relation

(4.88) $$\frac{2\{B_{(23\cdots r)}\}^2}{n_0n_1\prod\limits_{i=2}^{r}(n_i-1)}\frac{\partial f}{\partial B_{(23\cdots r)}}-\frac{2\{B_{(12\cdots r)}\}^2}{n_0n_1\prod\limits_{i=1}^{r}(n_i-1)}\frac{\partial f}{\partial B_{(12\cdots r)}}$$
$$+\frac{(-1)^{r-1}\{B_{(23\cdots r)}\}^2}{n_0n_1\prod\limits_{j=0}^{r}n_j\cdot\prod\limits_{i=2}^{r}(n_i-1)}\frac{\partial^2 f}{\partial\mu^2}-\frac{(-1)^{r}\{B_{(12\cdots r)}\}^2}{n_0n_1\prod\limits_{j=0}^{r}n_j\cdot\prod\limits_{i=1}^{r}(n_i-1)}\frac{\partial^2 f}{\partial\mu^2}$$
$$=\left[\frac{1}{n_0n_1}\left\{\frac{S_{(23\cdots r)}}{\prod\limits_{i=2}^{r}(n_i-1)}-\frac{S_{(12\cdots r)}}{\prod\limits_{i=1}^{r}(n_i-1)}\right\}-\sigma_{23\cdots r}\right]f.$$

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ON THE WAVE MOTION PROPAGATED ALONG AN ELASTIC CYLINDER WITH INFINITE LENGTH

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1. Introduction

The wave motion propagated along an elastic cylinder satisfying boundary conditions on its surface has been discussed already, but the equation to determine the velocity of the wave is so complex that only in a case when the radius of the cylinder is small, the velocity $c=\sqrt{E/\rho}$ is obtained approximately. If the approximation does not satisfied, what is the value of c ? If the radius is very large, the circumstance is similar with that of semi-infinite body bounded by a plane, and the wave may be similar with *Rayleigh wave*.

The writer discussed general case of this problem and obtained following result. Put

n = the number of frequency of the wave

a = the radius of the cylinder

$C_s=\sqrt{\mu/\rho}$

c = the velocity of the wave

$k = 2\pi na/C_s$,

then the relation between k and the velocity c of a plane wave propagated along the cylinder and not pure transversal wave is as illustrated in the Fig. 4, that is, c is gradually decreases from $\bar{C}$ $(=\sqrt{E/\rho}\,)$ to C_R (=velocity of Rayleigh wave) according to the increase of k. Moreover there appear 2nd value, 3rd value, ……… of c according to the increase of k.

Next, on the velocity of pure transversal plane wave propagated along the cylinder, $c=C_s$ is always possible. Moreover there appear 2nd value, 3rd value, ……… according to the increase of k.

2. Fundamental Formulae

Using the cylindrical coordinates (r, θ, z) we have

$$\left.\begin{aligned}
\rho\frac{\partial^2 s_r}{\partial t^2} &= (\lambda+2\mu)\frac{\partial\Delta}{\partial r} - \frac{2\mu}{r}\frac{\partial\omega_z}{\partial\theta} + 2\mu\frac{\partial\omega_\theta}{\partial z} \\
\rho\frac{\partial^2 s_\theta}{\partial t^2} &= (\lambda+2\mu)\frac{\partial\Delta}{\partial\theta}\frac{1}{r} - 2\mu\frac{\partial\omega_r}{\partial z} + 2\mu\frac{\partial\omega_z}{\partial r} \\
\rho\frac{\partial^2 s_z}{\partial t^2} &= (\lambda+2\mu)\frac{\partial\Delta}{\partial z} - \frac{2\mu}{r}\frac{\partial}{\partial r}(r\omega_\theta) + \frac{2\mu}{r}\frac{\partial\omega_r}{\partial\theta} \\
\rho\frac{\partial^2\Delta}{\partial t^2} &= (\lambda+2\mu)\left\{\frac{\partial^2\Delta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2} + \frac{1}{r}\frac{\partial\Delta}{\partial r} + \frac{\partial^2\Delta}{\partial z^2}\right\}
\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots (1),$$

where

$$\left.\begin{aligned} \boldsymbol{s} &= \text{displacement by strain} \\ \Delta &= \operatorname{div} \boldsymbol{s} \\ \boldsymbol{\omega} &= \frac{1}{2} \operatorname{rot} \boldsymbol{s} \end{aligned}\right\} \qquad (2)$$

and

$$\begin{aligned} \rho &= \text{density of cylinder} \\ \lambda, \mu &= \text{Lame's constants} \\ s_r, s_\theta, s_z &= r, \theta, z \text{ components of } \boldsymbol{s} \\ \omega_r, \omega_\theta, \omega_z &= r, \theta, z \text{ components of } \boldsymbol{\omega} \ . \end{aligned}$$

Now, let $\boldsymbol{P}_r$ be the stress which acts on unit area perpendicular to the radius r, and $P_{rr}, P_{r\theta}$ and P_{rz} be the r, θ and z components of $\boldsymbol{P}_r$ respectively, then we have

$$\left.\begin{aligned} P_{rr} &= \lambda\Delta + 2\mu \frac{\partial s_r}{\partial r} \\ P_{r\theta} &= \mu \left\{ \frac{\partial s_\theta}{\partial r} + \frac{1}{r} \frac{\partial s_r}{\partial \theta} - \frac{1}{r} s_\theta \right\} \\ P_{rz} &= \mu \left(\frac{\partial s_z}{\partial r} + \frac{\partial s_r}{\partial z} \right) \end{aligned}\right\} \qquad (3).$$

Now, suppose a plane wave

$$\left.\begin{aligned} s_r &= R(r)\, e^{i\kappa(z-ct)} \\ s_\theta &= \Theta(r)\, e^{i\kappa(z-ct)} \\ s_z &= Z(r)\, e^{i\kappa(z-ct)} \end{aligned}\right\} \qquad (4)$$

propagated along the z-axis. Substituting (4) into (1) gives

$$\frac{d^2D}{dr^2} + \frac{1}{r}\frac{dD}{dr} + p^2 D = 0 \qquad (5)$$

$$\frac{d^2Z}{dr^2} + \frac{1}{r}\frac{dZ}{dr} + s^2 Z = -i\kappa \frac{\lambda+\mu}{\mu} D \qquad (6)$$

$$i\kappa \frac{dZ}{dr} - s^2 R = \frac{\lambda+2\mu}{\mu} \frac{dD}{dr} \qquad (7)$$

$$\frac{d^2\Theta}{dr^2} + \frac{1}{r}\frac{d\Theta}{dr} + \left(s^2 - \frac{1}{r^2} \right)\Theta = 0 \qquad (8)$$

where

$$\left.\begin{aligned} D(r) &= \frac{dR}{dr} + \frac{1}{r}R + i\kappa Z \\ \triangle &= D(r)\, e^{i\kappa(z-ct)} \end{aligned}\right\} \qquad (9)$$

$$\left.\begin{array}{ll} C_p^2 = (\lambda + 2\mu)/\rho, & h = \kappa c/C_p \\ s^2 = \kappa^2\left(\dfrac{c^2}{C_s^2} - 1\right), & p^2 = \kappa^2\left(\dfrac{c^2}{C_p^2} - 1\right) \end{array}\right\} \cdots\cdots \quad (10).$$

At first, we can solve (8) with respect to $\Theta(r)$ independently of $R(r), Z(r)$ and $D(r)$, and get

$$\Theta(r) = AJ_1(rs) \cdots\cdots \quad (11)$$

as the solution which is finite at $r = 0$.

Next, solving (5), (6) and (7) simultaneously, we get

$$D(r) = BJ_0(rp) \cdots\cdots \quad (12)$$

$$\left.\begin{array}{l} Z(r) = CJ_0(rs) - \dfrac{i\kappa B}{h^2} J_0(rp) \\ R(r) = \dfrac{pB}{h^2} J_1(rp) - \dfrac{i\kappa C}{s} J_1(rs) \end{array}\right\} \cdots\cdots \quad (13)$$

as the finite solutions at $r = 0$.

If $s = 0$, taking the limits of (11) and (13), we get

$$\left.\begin{array}{l} \Theta(r) = Ar \\ Z(r) = \dfrac{-i}{\kappa} \dfrac{\lambda+2\mu}{\mu} BJ_0(rp) + E \\ R(r) = \dfrac{p}{\kappa^2} \dfrac{\lambda+2\mu}{\mu} BJ_1(rp) - \dfrac{1}{2} i\kappa Er \end{array}\right\} \cdots\cdots \quad (14).$$

Where A, B, C and E are some constants and $J_0(Z)$ and $J_1(Z)$ are Bessel's functions. Moreover these solutions must satisfy some boundary conditions on the surface of medium. Let the wave propagate along a cylinder whose axis is in the z-direction, radius is a, then the boundary conditions are

$$P_{rr} = 0, \quad P_{r\theta} = 0, \quad P_{rz} = 0 \quad at\ r = a.$$

3. Solution satisfying $P_{r\theta} = 0$ at $r = a$.

From the given condition, we get

$$\left[\frac{d\Theta}{dr} - \frac{1}{r}\Theta\right]_{r=a} = 0 \cdots\cdots \quad (15).$$

Substituting (11) into (15) gives $AJ_2(as) = 0$, and assuming $A \neq 0$, we get

$$J_2(as) = 0 \cdots\cdots \quad (16).$$

Therefore as is the zero points of $J_2(Z)$ and putting

$$\left.\begin{array}{lll} b_0 = 0, & b_1 = 5.135, & b_2 = 8.417 \\ b_3 = 11.620, \cdots\cdots & & \end{array}\right\} \cdots\cdots \quad (17)$$

gives

$$as = b_j \quad (j = 0, 1, 2, \cdots\cdots) \qquad \cdots\cdots (18)$$

Now, let n be the number of frequency of the wave, then

$$\kappa = \frac{2\pi n}{c} \qquad \therefore \quad s^2 = \frac{4\pi^2 n^2}{C_s^2}\left(1 - \frac{C_s^2}{c^2}\right)$$

and putting

$$k = \frac{2\pi n a}{C_s} \qquad \cdots\cdots (19)$$

gives

$$c = \frac{1}{\sqrt{1 - \frac{b_i^2}{k_2}}} C_s \qquad \cdots\cdots (20).$$

Here,

$$k > b_j \qquad \cdots\cdots (21)$$

must be satisfied so as to the value of c exists otherwise c becomes imaginary.

The formula (11) is the solution of $\Theta(r)$ provided that the value of c is determined by (20). Hence, we have following conclusion by above description:

when

$$b_N < k < b_{N+1} \qquad \cdots\cdots (22)$$

the values of the velocity c exist $N+1$, *i.e.*,

$$c = C_0 (= C_s), \quad C_1, \ C_2, \cdots\cdots\cdots, C_N$$

and for each value of c, we get

$$\left.\begin{aligned} s_\theta &= Ar\, exp \cdot 2n\pi i\left(\frac{1}{C_s} Z - t\right) \\ s_\theta &= AJ_1\left(\frac{b_j}{a} r\right) exp \cdot 2n\pi i\left(\frac{1}{C_j} Z - t\right) \\ j &= 1, 2, 3, \cdots\cdots, N \end{aligned}\right\} \qquad \cdots\cdots (23).$$

For example, when $a = 1^{cm}$, $C_s = 3\times 10^5 cm/sec$, from (21) we have

$$n > \frac{C_s}{2\pi a} b_j = 4.77\times 10^4 \ b_j/sec$$

and

$$4.77\times 10^4 \ b_1 = 2.45\times 10^5$$

therefore if $n < 2.45\times 10^5/sec$ we get only one value of c, i.e., $c = C_s$ and

$$\Theta(r) = Ar\,exp\cdot 2n\pi i\left(\frac{1}{C_s}Z - t\right).$$

If $4.77\times 10^4\ b_2 = 4.01\times 10^5 < n < 5.54\times 10^5 = 4.77\times 10^4\ b_3$,

we have three values of c (i. e. C_s, C_1 and C_2) and also the solutions of $\Theta(r)$ can exist three, e. g., for $n = 5\times 10^5/sec$ we have

$$\left\{1 - \left(\frac{b_1}{k}\right)^2\right\}^{-\frac{1}{2}} = 1.15, \quad \left\{1 - \left(\frac{b_2}{k}\right)^2\right\}^{-\frac{1}{2}} = 1.67,$$

hence three values of c, i. e.,

$$C_0 = C_s = 3\times 10^5\ cm/sec$$
$$C_1 = 1.15\ C_s = 3.45\times 10^5\ cm/sec$$
$$C_2 = 1.67\ C_s = 5.02\times 10^5\ cm/sec$$

can exist, and corresponding s_θ are

$$s_{\theta 0} = A_0 r\,exp\cdot 2n\pi i\left(\frac{1}{C_s}Z - t\right)$$

$$s_{\theta 1} = A_1 J_1\left(\frac{b_1}{a}r\right) exp\cdot 2n\pi i\left(\frac{1}{C_1}Z - t\right)$$

$$s_{\theta 2} = A_2 J_1\left(\frac{b_2}{a}r\right) exp\cdot 2n\pi i\left(\frac{1}{C_2}Z - t\right)$$

respectively.

4. Solutions satisfying $P_{rr} = 0$ and $P_{rz} = 0$ at $r = a$.

From the given conditions we get

$$\left.\begin{aligned} &\left[\lambda D(r) + 2\mu\frac{d}{dr}R(r)\right]_{r=a} = 0 \\ &\left[\frac{d}{dr}Z(r) + i\kappa R(r)\right]_{r=a} = 0 \end{aligned}\right\} \qquad (24).$$

Substituting (13) into (24) gives

$$\left.\begin{aligned} &\frac{1}{h^2}\left\{(s^2-\kappa^2)J_0(ap) - \frac{2p}{a}J_1(ap)\right\}B \\ &\qquad + 2i\kappa\left\{\frac{1}{as}J_1(as) - J_0(as)\right\}C = 0 \\ &\frac{2i\kappa p}{h^2}J_1(ap)B + \frac{\kappa^2 - s^2}{s}J_1(as)\,C = 0 \end{aligned}\right\} \qquad (25)$$

and we get

$$\begin{vmatrix} \frac{1}{h^2}\left\{(s^2-\kappa^2)J_0(ap)-\frac{2p}{a}J_1(ap)\right\}, & 2i\kappa\left\{\frac{1}{as}J_1(as)-J_0(as)\right\} \\ \frac{2i\kappa p}{h_2}J_1(ap), & \frac{\kappa^2-s^2}{s}J_1(as) \end{vmatrix}=0,$$

that is

$$\begin{aligned} a(s^2-\kappa^2)^2 J_0(ap)J_1(as) & -2p(s^2+\kappa^2)J_1(ap)J_1(as) \\ & +4\kappa^2 aps\, J_1(ap)J_0(as)=0 \end{aligned} \quad \cdots\cdots\cdots\cdots \quad (26),$$

in order to that at least one of B and C is not zero.
We can determine the value of c from (26) and then $B:C$ from (25), and get

$$\left.\begin{aligned} R(r) &= Qp\ \{(\kappa^2-s^2)J_1(as)J_1(rp)-2\kappa^2\ J_1(ap)J_1(rs)\} \\ Z(r) &= i\kappa Q\ \{(s^2-\kappa^2)J_1(as)J_0(rp)-2ps\,J_1(ap)J_0(rs)\} \end{aligned}\right\} \quad \cdots\cdots\cdots \quad (27),$$

where Q is some constant.
Putting

$$\left.\begin{aligned} & J_1(Z)=\frac{Z}{2}X_1(Z) \\ & C_s^2/c^2=\xi, \qquad C_s^2/C_p^2=\mu/(\lambda+2\mu)=\gamma \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (28)$$

we can rewrite (26) and (27) as follows,

$$\begin{aligned} & (2\xi-1)^2 X_1(k\sqrt{1-\xi})J_0(k\sqrt{\gamma-\xi}) \\ & \quad -(\gamma-\xi)X_1(k\sqrt{\gamma-\xi})X_1(k\sqrt{1-\xi})+4\xi(\gamma-\xi)X_1(k\sqrt{\gamma-\xi})J_0(k\sqrt{1-\xi})=0 \end{aligned} \quad \cdots\cdots \quad (29)$$

$$\left.\begin{aligned} R(r) &= Uk(\gamma-\xi)\,r\Big[(2\xi-1)X_1(k\sqrt{1-\xi})X_1\left(\frac{r}{a}k\sqrt{\gamma-\xi}\right) \\ & \qquad -2\xi X_1(k\sqrt{\gamma-\xi})X_1\left(\frac{r}{a}k\sqrt{1-\xi}\right)\Big] \\ Z(r) &= 2iaU\sqrt{\xi}\Big[-(2\xi-1)X_1(k\sqrt{1-\xi})J_0\left(\frac{r}{a}k\sqrt{\gamma-\xi}\right) \\ & \qquad -2(\gamma-\xi)X_1(k\sqrt{\gamma-\xi})J_0\left(\frac{r}{a}k\sqrt{1-\xi}\right)\Big] \end{aligned}\right\} \quad \cdots\cdots \quad (30)$$

where U is some constant.

Let $f(\xi)$ be the left side of (29), and find the positive root of $f(\xi)=0$, and substitute it into (30), then we get the required solutions of $R(r)$ and $Z(r)$. The value of c is determined from (28).

5. When k is small.

In this case, putting

$$J_0(k\sqrt{\gamma-\xi})=1, \quad X_1(k\sqrt{1-\xi})=1 \quad \text{etc., we get}$$

$$\xi=\frac{1-\gamma}{3-4\gamma} \qquad (31)$$

from (29) and

$$c=\frac{1}{\sqrt{\xi}}C_s=\sqrt{\frac{E}{\rho}} \qquad (32)$$

where

$$E=\frac{(3\lambda+2\mu)\mu}{\lambda+\mu} \qquad (33).$$

Of course, these are well known result. From (30) we get

$$\left.\begin{aligned} R(r)&=k(1-2\gamma)Wr \\ Z(r)&=2ai\sqrt{(1-\gamma)/(3-4\gamma)}\cdot W \end{aligned}\right\} \qquad (34)$$

where W is some constant. The real parts of s_r, s_z corresponding to (34) are

$$\left.\begin{aligned} s_r&=k(1-2\gamma)Wr\cos\kappa(Z-ct) \\ s_z&=-2a\sqrt{(1-\gamma)/(3-4\gamma)}W\sin\kappa(Z-ct) \end{aligned}\right\} \qquad (35).$$

6. General case.

Putting

$$\bar{C}=\sqrt{E/\rho} \qquad (36)$$

gives

$$C_s=\sqrt{\frac{1-\gamma}{3-4\gamma}}\cdot\bar{C}, \quad c=\sqrt{\frac{1}{\xi}\cdot\frac{1-\gamma}{3-4\gamma}}\,\bar{C} \qquad (37).$$

Table 1. (from Shiba, Iida)

	E 10^6kg·wt·/cm^2	μ 10^6kg·wt·/cm^2	$\gamma=\frac{3\mu-E}{4\mu-E}$
Ag	0.805	0.294	0.21
Al	0.719	0.272	0.26
Cu	1.250	0.464	0.23
Ni	2.054	0.785	0.28
wrought-iron	2.17	0.83	0.28
brass	1.05	0.43	0.36
bronze	0.824	0.350	0.39
granite (Yahagi Aichi)	0.713	0.304	0.39
andesite (Odagiri Nagano)	0.683	0.287	0.38

Now, we have the relation

$$\gamma = \frac{3\mu - E}{4\mu - E} \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (38)$$

and the values of μ, E and γ of several materials are shown in the table 1. In this table, the value of γ lays near in the interval (0.2, 0.4).

At first, we calculate the root of $f(\xi)=0$ for various values of k when $\gamma=0.2$, 0.3 and 0.4, and then find the values of $c/\bar{C}$ from (37), and $R(r), Z(r)$ from (30).

When $\gamma=0.3$.

In this case $\sqrt{(1-\gamma)/(3-4\gamma)}=0.6236$, hence

$$C_s \doteqdot 0.624\bar{C}, \quad c \doteqdot \frac{1}{\sqrt{\xi}} \times 0.624\bar{C}.$$

(a) for $k=0$.

By preceding section, $c=\bar{C}$.

(b) for $k=1$.

The graph of $f(\xi)$ is shown in the Fig. 1, and it indicates that the equation $f(\xi)=0$ has only one real root about 0.4 in the interval (0, 1.4), and it is found that more precise value of this root is 0.396 by numerical calculation.

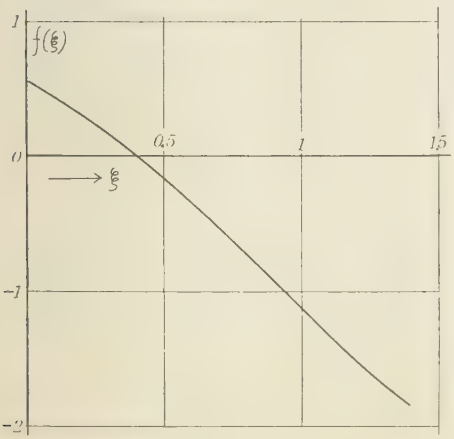


Fig. 1 The graph of $f(\xi)$ when $\gamma=0.3$ and $k=1$.

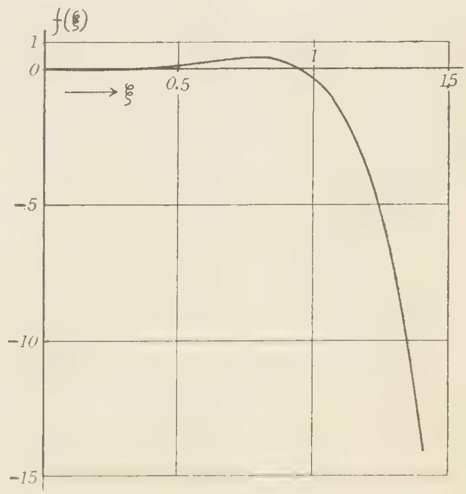


Fig. 2 The graph of $f(\xi)$ when $\gamma=0.3$ and $k=4$.

(c) for $k=4$.

Similarly as above, we get Fig. 2 and $\xi=0.029$, 0.228, 0.958 as the roots of $f(\xi)=0$, and then $c=3.66\bar{C}$, $1.31\bar{C}$, $0.64\bar{C}$.

Similarly, we get the values of ξ and $c/\bar{C}$ for various values of k and the table 2 shows its result.

Table 2.

k	$C/\bar{C}$ when $\gamma=0.3$		
0	1.00		
1	0.99		
2	0.95		
3	0.79		
3.5	0.70		
4	0.64	1.31	3.66
5	0.60	1.10	2.13
7	0.60	0.94	1.26
9	0.58	1.11	1.35

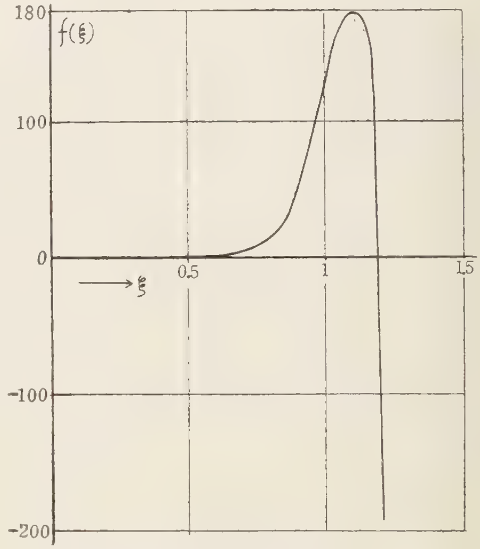


Fig. 3 The graph of $f(\xi)$ when $\gamma=0.3$ and $k=0.9$.

Let C_R be the velocity of Rayleigh wave, then the calculation shows that $C_R \doteqdot 0.58\bar{C}$, hence the first value of $c/\bar{C}$ for $k=9$ in the table 2 indicates that $c=C_R$.

Now, as obvious by Fig. 2 and Fig. 3 etc., the value of $|f'(\xi)|$ is very small when ξ is near the corresponding values of the 2^{nd}, $3^{rd}\cdots\cdots$ values of c, and $f(\xi)$ is nearly always equal to zero. Hence the positive root of $f(\xi)=0$ greatly changes its value by small variation of the values of γ and k. Therefore the energy of wave corresponding to these velocity will be small even if it exists.

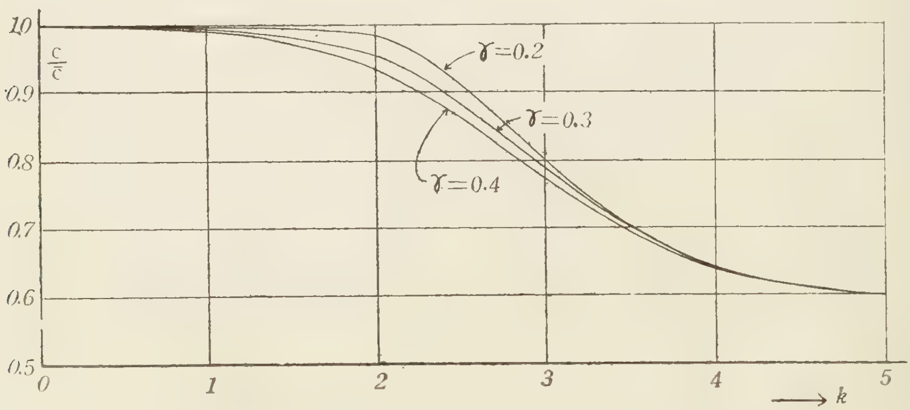


Fig. 4 The relation between $c/\bar{C}$ and k, where c is the 1^{st} value of the velocity of wave.

Again, we did similar calculations for $\gamma=0.2$ and $\gamma=0.4$ and Fig. 4 illustrates their result.

7. Conclusion

As obvious from section 3 and 4, s_θ is independent of s_r and s_z, while s_r and s_z are dependent with each other. The first value of c concerning s_θ is always equal to C_s, and there appear 2^{nd}, 3^{rd} ………… values of c according to the increase of k from 5.135. Regarding s_r and s_z, as obvious from Fig. 4, the first value of c is as follows;

$c \doteqdot \bar{C}$ ………… when $0 \leqq k \lessapprox 1$

c decreases from $\bar{C}$ to C_R when k increases from $k \doteqdot 1$ to $k \doteqdot 5$

$c \doteqdot C_R$ ………… when $5 \lessapprox k$

e. g., when $a=2^{cm}$ and $C_s=3\times10^5 cm/sec$, we have $k \doteqdot n/2.4\times10^4 sec$, hence, if n is in order that of sound wave, we get $k<1$ and $c \doteqdot \bar{C}$, while if $n=100^{K.C}/sec$ which is in order that of supersonic sound wave, we get $k \doteqdot 4.2$ and $c \doteqdot 0.65\bar{C}$.

It is noticeable that the variation of the value of c caused by the variation of γ is very small and mainly depends on the variation of k.

ON THE BALLOONING PROBLEM (III)
(Problems of Inner ballooning)

Akira ONUKI

(Received July 15, 1959)

7. Fundamental equations of inner ballooning.

In this section we will treat the problems of inner ballooning, that is, the balloon of the rotating yarn inside the cap around the bobbin. As shown in Fig. 5, at the point A comes the yarn in the cap and is wund to the bobbin at the point D. The part AD can be considered to lie on a horizontal plane, provided we neglect the gravitational force mg, for all remaining forces are acting horizontally upon the thread.

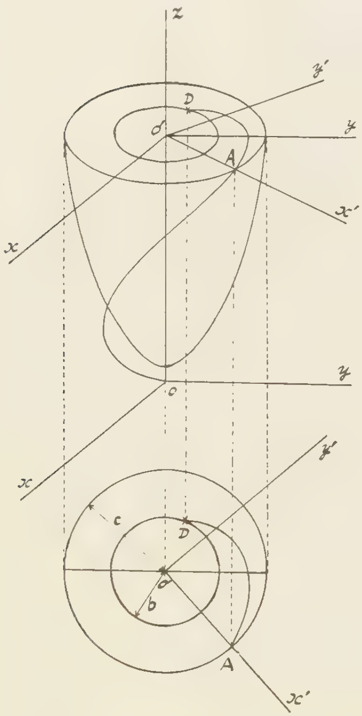


Fig 5.

We denote by c and b the radii of cap and the bobbin respectively. For convenience we take new coordinate axis having common origin and z axis with the old coordinate, but its x-axis being taken through the point A. The polar coordinate r, θ with respect to this new axis completely specifies the shape of yarn, hence, it is unnecessary, and also impossible, to use z as an independent variable as in the outer ballooning calculation.

Now from the fundamental eqs in § 1. *) It can be obtained easily that

$$\frac{d}{ds}\left(T+\frac{1}{2}m\omega^2r^2\right)=P\omega^2r^2\sin\tau\cos\tau \quad \cdots\cdots (7.1)$$

$$\frac{d}{ds}(Tr^2\dot{\theta})=P\omega^2r^3\sin\tau+m\omega v\frac{d}{ds}(r^2) \quad \cdots (7.2)$$

$$r\dot{\theta}=\cos\tau \quad \cdots\cdots (7.3)$$

$$\dot{r}^2+r^2\dot{\theta}^2=1 \quad \cdots\cdots (7.4)$$

where $\cdot$ denotes $\frac{d}{ds}$, dsd T means the-modified tension, i. e.

$$T \pm mv^2$$

*) Kumamoto J. of Sci. Vol. 1, p. 39

From (7.3), and (7.4) it becomes

$$\dot{r} = \mp \sin\tau. \quad\cdots\cdots (7.5)$$

At present we have to take upper minus sign, because from the definition of angle τ, $\sin\tau$ is always plus and r is diminishing from the cap edge to the bobbin.

Hence it follows that

$$\frac{d\theta}{dr} = -\frac{\cot\tau}{r}. \quad\cdots\cdots (7.6)$$

Using (7.5), (7.1) and (7.2) become

$$\frac{d}{dr}\left(T + \frac{1}{2} m\omega^2 r^2\right) = -P\omega^2 r^2 \cos\tau \quad\cdots\cdots (7.7)$$

$$\frac{d}{dr}\left(Tr\cos\tau - m\omega v r^2\right) = -P\omega^2 r \quad\cdots\cdots (7.8)$$

which are, together with (7.6), the required fundamental equations for inner ballooning.

Here we are taking r as independent variable, with whom, T, τ, θ must be expressed. In order to see clearly what is the difference of the equations between outer and inner ballooning, we rewrite here the equations of outer ballooning written by r and θ in the same way.

$$\left(\sin^2\tau - \frac{T_0^2\cos^2\alpha_0}{T^2}\right)^{\frac{1}{2}} \frac{d}{dr}\left(Tr\cos\tau - m\omega v r^2\right) = \mp P\omega^2 r^3 \sin\tau$$

$$\left(\sin^2\tau - \frac{T_0^2\cos^2\alpha_0}{T^2}\right)^{\frac{1}{2}} \frac{d}{dr}\left(T + \frac{1}{2} m\omega^2 r^2\right) = \mp P\omega^2 r^2 \sin\tau\cos\tau$$

$$\left(\sin^2\tau - \frac{T_0^2\cos^2\alpha_0}{T^2}\right)^{\frac{1}{2}} \frac{d\theta}{dr} = \mp \frac{\cos\tau}{r}$$

$$\frac{dr}{dz} = \left(\frac{T^2\sin^2\tau}{T_0^2\cos^2\alpha_0} - 1\right)^{\frac{1}{2}}.$$

Therefore, when α_0 becomes $\pi/2$, then z becomes zero, and above equations coincide with those of (7.6), (7.7), (7.8).

This explains clearly why we had taken r as independent variable.

Returning to (7.6), (7.7), (7.8) again, we put

$$r = c\rho \qquad T = T_1 T_A \quad\cdots\cdots (7.9)$$

where T_A is the tension of yarn at the point A.*)

The equations then become

$$\frac{d\theta}{d\rho} = -\frac{\cot\tau}{\rho} \quad\cdots\cdots (7.10)$$

*) Note that the definition of T_1 differs from those of § 1.

$$\frac{d}{d\rho}(T_1\rho cos\tau) = 2\mu\rho - \gamma'\rho^3 \quad \cdots\cdots (7.11)$$

$$\frac{dT_1}{d\rho} = -\beta'\rho - \gamma'\rho^2 cos\tau \quad \cdots\cdots (7.12)$$

where

$$\mu = \frac{m\omega vc}{T_A} \quad \cdots\cdots (7.13)$$

$$\gamma' = \frac{P\omega^2 c^3}{T_A} \quad \cdots\cdots (7.14)$$

$$\beta' = \frac{m\omega^2 c^2}{T_A} \ . \quad \cdots\cdots (7.15)$$

It should be commented here that no term except those of gravitational force had been neglected in the derivation of above equations. As one, sees, these have far simpler form than those of the outer ballooning, however, it seems to be difficult to solve them exactly. Being our immediate purpose of computation is mainly to know the tension around the bobbin, we confine ourselves to solve them only approximately.

8. No-air-drag solution.

Like in the case of outer ballooning, we first look for a solution when both of the air drag and the sending velocity can be neglected.

Putting $\mu=0, \gamma'=0$, then the equations become

$$\frac{d\theta}{d\rho} = -\frac{cot\tau}{\rho} \quad \cdots\cdots (8.1)$$

$$T_1\rho cos\tau = cos\tau_A \quad \cdots\cdots (8.2)$$

$$T_1 = 1 + \frac{\beta'}{2}\cdot(1-\rho^2) \quad \cdots\cdots (8.3)$$

in consideration of the boundary condition. Hence, it follows that

$$tan\tau = (T_1^2\rho^2 sec^2\tau_A - 1)^{\frac{1}{2}}$$

and θ can be integrated as

$$\theta = \int_\rho^1 \frac{d\rho}{\rho(\rho^2 T_1^2 sec^2\tau_A - 1)^{\frac{1}{2}}} \ . \quad \cdots\cdots (8.4)$$

Inserting to the T_1 of this equation the value from (8.3), (8.4) we obtain the required relation $\rho=\rho(\theta)$ which determines the shape of balloon. Integral in the right side will be expressed by the elliptic function like in the case of no-air-drag outer ballooning, but the use of the familiar numerical integration may be more practicable.

Let us denote the values of each variable at the point A and D by adding the suffix A and D respectively, then it must hold from above equations

$$\left.\begin{aligned} T_{1D}\,\rho_D &= \cos\tau_A \\ T_{1D} &= 1+\frac{\beta'}{2}(1-\rho_D^2) \end{aligned}\right\} \qquad (8.5)$$

hence, it follows that

$$\cos\tau_A=\rho_D\left\{1+\frac{\beta'}{2}(1-\rho_D^2)\right\} \qquad (8.6)$$

giving the value of τ at A from the known β' and $\rho_D=b/c$, while θ_D is given by

$$\theta_D=\int_{\rho_D}^{1}\frac{d\rho}{\rho(\rho^2T_1^2\sec^2\tau_A-1)^{\frac{1}{2}}}. \qquad (8.7)$$

Thus our process of computation becomes as follows

firstly, determine τ_A from (8.6)

secondly, from (8.3) obtain function $T_1(\rho)$

thirdly, from (8.4) obtain $\rho=\rho(\theta)$

and finally from (8.2) obtain the value of τ for arbitrary ρ.

These process of the computation will supply all information about the inner ballooning, therefore, the question of which has been completely solved in the case of no-air-drag.

Occasionally it has been questioned in the practical case that how T_D, namely the tension of wund yarn around the bobbin, changes according to the diameter's variation of the bobbin. When the bobbin diameter b slightly increases from empty state to full state, ω also increases according to the formula

$$\omega=\omega_b-v/b$$

T_D rewritten by original notation from (8.5), is given by

$$T_D=T_A+\frac{mv^2}{2}\left(\frac{c\omega_b}{v}-\frac{c}{b}\right)^2\left(1-\frac{b^2}{c^2}\right). \qquad (8.8)$$

In the following table (Table 9) T_D-T_A is tabulated against b/c, which increases from some 0.4 (empty bobbin) to some 0.7 (full bobbin), in case when we adopt the most practical values for other constants. One sees in this example that the difference of the tension is only 0.3 gr. weight in the case of 300 denier.

It must be noted that T_D-T_A takes the maximum when $\frac{b}{c}$ takes the value $\left(\frac{c\omega_b}{v}\right)^{-\frac{1}{3}}$.

Table 9.

v = 600 m/min. = 1.10^3
ω_b = 5750 r.p.m. = 6.10^2
$m_{75}=0.833\ 10^{-4}$ $m_{300}=3.333\ 10^{-4}$
$c=7.8$

b/c	$m=75$ T_D-T_A(gr)	$m=300$ T_D-T_A(gr)
empty 0.4	0.17	0.68
0.5	0.23	0.91
0.6	0.25	0.98
full 0.7	0.23	0.91

§ 9. General solution of the inner ballooning.

Proceeding to more general case, in which none of the terms are neglected, first

we obtain from (7.11)

$$T_1\rho\cos\tau = \cos\tau_A - \mu(1-\rho^2) + \frac{\gamma'}{4}(1-\rho^4) \quad \cdots\cdots (9.1)$$

and from (7.12)

$$T_1\frac{dT_1}{d\rho} = -\beta'\rho\left(T_1 + \frac{\gamma'}{\beta'}\cos\tau_A\right) + \mu\gamma'\rho(1-\rho^2) - \frac{\gamma'}{4}\rho(1-\rho^4). \quad \cdots\cdots (9.2)$$

This is a differential equation of T_1 for ρ as an independent variable. Unfortunately this cannot be integrated directly, unless one uses the method of numerical integration. However, naturally one might suppose that $\gamma'\mu \ll \beta'$ $\gamma'^2 \ll \beta'$ and therefore, provided these assumptions hold in the actual state, the equations can be written in the form

$$T_1\left(T_1 + \frac{\gamma'}{\beta'}\cos\tau_A\right)^{-1} dT_1 = -\beta'\rho d\rho$$

and easily be integrated as

$$T_1 - \frac{\gamma'}{\beta'}\cos\tau_A \log\left\{\frac{T_1 + \frac{\gamma'}{\beta'}\cos\tau_A}{1 + \frac{\gamma'}{\beta'}\cos\tau_A}\right\} = 1 + \frac{\beta'}{2}(1-\rho^2) \quad \cdots\cdots (9.3)$$

or in the same approximation,

$$T_1 = 1 + \frac{\beta'}{2}(1-\rho^2) + \frac{\gamma'}{\beta'}\cos\tau_A \log\left\{1 + \frac{\beta'}{2}(1-\rho^2)\right\} \quad \cdots\cdots (9.4)$$

from which it follows

$$T_D = T_A + \frac{m\omega^2c^2}{2}\left(1 - \frac{b^2}{c^2}\right) + \frac{cPT_A\cos\tau_A}{m}\log\left\{1 + \frac{m\omega^2c^2}{2T_A}\left(1 - \frac{b^2}{c^2}\right)\right\} \quad \cdots\cdots (9.5)$$

$$\cos\tau = \frac{\cos\tau_A}{T_1\rho} \quad \cdots\cdots (9.6)$$

$$\theta = \int_\rho^1 \frac{d\rho}{\rho(T_1^2\rho^2\sec^2\tau_A - 1)^{\frac{1}{2}}} \quad \cdots\cdots (9.7)$$

$$\theta_D = \int_{\rho_D}^1 \frac{d\rho}{\rho(T_1^2\rho^2\sec^2\tau_A - 1)^{\frac{1}{2}}} \quad \cdots\cdots (9.8)$$

and

$$\cos\tau_A = \frac{\rho_D + \left(\frac{\beta'}{2}\rho_D + \mu\right)(1-\rho_D^2) - \frac{\gamma'}{4}(1-\rho_D^4)}{1 - \frac{\gamma'}{\beta'}\rho_D \log\left\{1 + \frac{\beta'}{2}(1-\rho_D^2)\right\}}. \quad \cdots\cdots (9.9)$$

The process of computation is almost the same as before, namely, obtain τ_A from (9.9) and $T_1(\rho)$ from (9.4), then τ from (9.6) and θ from (9.7) for arbitrary ρ. The tables and figures in the end of this section will show the results of calculation proceeded in this way.

Table 10.

	Empty bobbin		Full bobbin	
	m_{75}	m_{300}	m_{75}	m_{300}
ρ_D	0.413	(0.413)	0.705	(0.705)
β'	0.0406	0.1620	0.0866	0.3463
μ	0.0189	0.0753	0.0275	0.1099
γ'	0.0494	0.0494	0.1056	0.1086
γ'/β'	1.22	0.305	1.22	0.305
τ_A	64° 41′	60° 19′	43° 18′	35° 15′
$\cos\tau_A$	0.428	0.495	0.728	0.817

The values of constant.
$\omega = 3.10^3$ R.P.M. $T_A = 10$ gr. $v = 600$ m/min.

Table 11.

	Empty bobbin				Full bobbin			
ρ	m_{75} θ	T_1	m_{300} θ	T_1	m_{75} θ	T_1	m_{300} θ	T_1
0.450	44°38′	1.025						
0.475	38°10′	1.024	54°09′	1.072				
0.500	33°14′	1.025	44°24′	1.070				
0.600	19°25′	1.020	24° 0′	1.059				
0.700	12°05′	1.016	14°47′	1.047				
0.750					26°08′	1.036	42°07′	1.095
0.775					21°39′	1.032	31° 0′	1.086
0.800	6°39′	1.011	8°19′	1.034	17°56′	1.029	24°42′	1.078
0.900	2°59′	1.006	3°40′	1.018	7°17′	1.015	9°44′	1.041
1.000		1.000		1.000		1.000		1.000

Relation of $\rho - \theta$, T_1 where $T_1 = T/T_{iA}$
T_{iA} = tension at A of inner balloon and $\rho = r/c$

A figure of typical example of the inner balloon is shown in Fig. 6 on the next page. It shows how the angle τ_A is changed according to the b/c. The curve is very nearly to a straight line, owing to relatively high tension than in the case of the outer balloon.

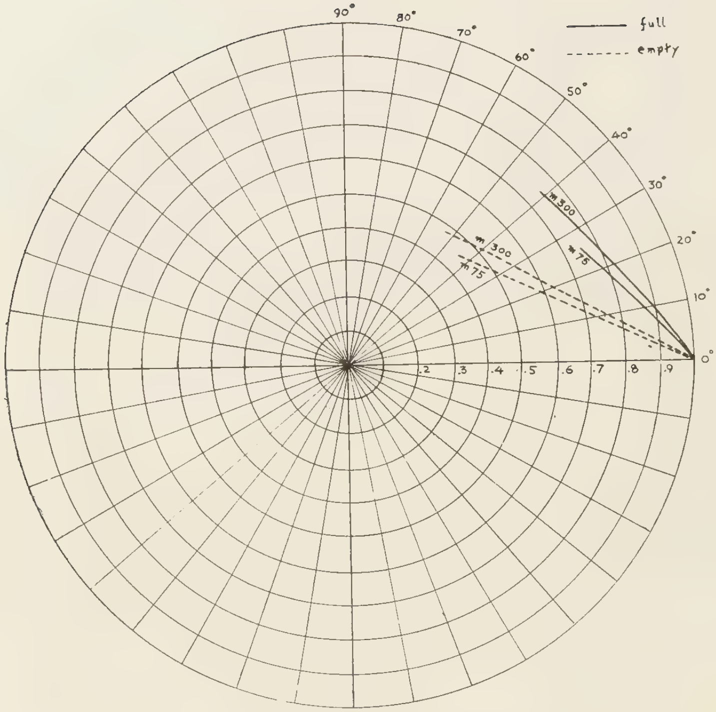


Fig 6.

Shape of the inner balloon.

THE 1958 ACTIVITY OF THE ASO VOLCANO

Hatao MATSUMOTO and Seizo TANAKA

(Received July 15,1959)

Introduction

The Aso Volcano is famous in the world for its gigantic caldera and its activity. The active central cone is called Naka-dake (Centre Peak), which rises from the center part of the caldera to an altitude of 1500 m.

As shown in the sketch map, Naka-dake with seven-fold surrounding craters has four active centers which are called No. 1, No. 2, No. 3 and No. 4 from the north at present, and erupt alternately. It erupted in 1958 after 25 years of quiescence except for an occasional explosion.

This paper describes the explosion products and eruption ejectas briefly. The writers wish to express their sincere thanks to Prof. I. Iwasaki, Dr. T. Katsura, Mr. T. Ozawa, and Mr. M. Yoshida of the Tokyo Institute of Technology, and Mr. M. Yamaguchi of the Kyusyu University for their kind helps and suggestions in preparing this manuscript. Their thanks are also extended to Mr. M. Kamada of the Kagoshima University for his valuable advice during this study.

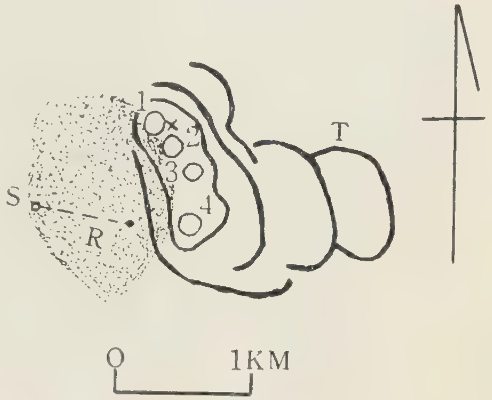


Fig. 1 The sketch map near the crater.
T; The highest peak of Naka-dake.
R; Rope way.
S; Shops and the bus termial.
1–4; Number of craters at present.
X; Gas sampling point.
Spot marks; The distribution of the ejectas in June 24, 1958.

Eruption of 1958

Condition before Explosion: Nake-dake began its activity in the last ten days of Oct., 1957 and exploded in a small scale in Nov. 2, Dec. 23, 1957 and April 5, 1958, hurling out ash or black smoke. From the second ten days of April, 1958, the crater became quiet; only a few small fumaroles of the east side of the crater wall and the crater emitted steam or sulphur, and the crater bottom looked like a dry ash pan or a small boiling pond.

Beginning of Activity: The 1958 eruption began with big explosion occurred at about 10:15 p. m. in June 24, without prior seismic or other warning. A seismometer

of the Aso Meteorological Observatory, however, recorded the amplitude of this explosion as 84 microns in maximum only 2–3 seconds before its activity. At this sudden explosion, about 50,000 tons of old ejectas, which had been packed in the vent, were erupted and volcanic ash was flung to the distance of about 13 km. to the southwestern caldera wall. Twelve people were killed by this explosion. The ejected ash was enormous in amount and the deposit reached to about 70 cm. in thicknes at the west side of the crater. The other ejectas are old lava, volcanic breccia, volcanic agglomerate and volcanic tuff, which had been cemented in the vent. The largest ejecta

Fig. 2. The largest ejecta of volcanic agglomerate which was packed in the vent and was erupted as far as about 200 m. from the No. 1. crater by the explosion in June 24.

is volcanic agglomrate which is about 5 tons in weight as shown in Fig. 2. Almost all of the ejectas were thrown out toward shelters and shops. Blocks about 0.3 m. in diameter were thrown as far as 2 km.

After this terrible explosion, the crater became quiet again. However, this phenomenon was not a simple explosion but an omen of an eruption after the cease of twenty-five years.

Description of the eruption: An eruption started in Sep. 29, 1958, and continued until Nov. 7. Eruptions occurred at intervals of 2–15 seconds, most commonly at about 6 seconds. The ejecta ranged from ash to great masses of molten lava which separated into blocks on flight, some as large as 2 m. in length. They were thrown as high as 400 m., and the eruption showed a scence like a beautiful fire-flower at night as shown in Fig. 3. During this activity, a seismograph at the Aso Meteorological Observatory continued to record an amplitude of about 2–4 microns.

Ejectas

Ejectas of the Explosion

Fig. 3. The eruption is like a beautiful fire-flower in Oct. 30. S. S. F. 4., time 30 sec.

The ejectas of the explosion in June 24 were former lava, volcanic bomb, lapilli, sand, and ash, which had been packed or cemented in the vent, and we could not find new lava or ejectas.

1) Old lava (Olivine bearing two pyroxene andesite)

This is generally a dark compact, porphyritic rock with abundant phenocrysts of plagioclase and has flat cavities. Under the microscope, phenocrysts of plagioclase, augite, hypersthene, and olivine are unevenly scattered in the fine-grained groundmass.

Phenocrysts: Plagioclase is idiomorphic or xenomorphic in crystal, and can be divided into two types. One is fresh without any inclusions, and the other has numerous inclusions, showing magmatic reaction. Zonal structure is very pronounced; the composition varies from An 85 to An 46, but the core is always An 55–60. The index of plagioclase glass after Foster's experiment[1]. $n=1.541$, and An 62.

Most of augite is rounded in form and shows very weak pleochroism, and the optical data are as follows. $\alpha=1.688$, $\beta=1.691$–1.694. $(+)\,2V=48°$–$50°$. Hypersthene is idiomorphic and always pleochroic. The optical property is as follows. $\beta=1.695$–1.698. $(-)\,2V=60°$. It is common that hypersthene is surrounded by augite which shows $(+)\,2V=50°$. Olivine is idiomorphic or hypidiomorphic in crystal and is rarely corroded. It does not show any reaction rim. $\beta=1.688$–1.692. $2V=(+)86°$–$(-)88°$.

Groundmass: It is compact, fine-grained, consisting of plagioclase, hypersthene $(-)\,2V=65°$, augite $(+)\,2V=10°$–$40°$ and magnetite, and filled by alkali-feldspar, anhydrite and alunite in cavities.
This rock contains pegmatitic fragments as xenolith. Anhydrite is colourless with well developed cleavages in the directions of three pinacoid. Optical data are as follows. $(+)\,2V=42°$–$43°$. $\alpha=1.573$, $\beta=1.577$, $\gamma=1.612$, $\gamma-\alpha=0.039$.

2) Andesitic tuffite

The rock is grey or red in color and porous, and consists of volcanic ash. In general, fine sublimate sulphur adheres on surface or in small druses. Optical prope-

rties of the constituent minerals are as follows.
plagioclase; An 47–60, glass after fusion; n=1.544, An 65.
hypersthene; (−) 2V=66°–69°. olivine; (−) 2V=86°–88°.
augite; (+) 2V=56°–58°. β=1.689–1.697. augite in reaction rim, (+) 2V=56°.
alunite; ω=1.572, ε=1.588.

3) Tuff

This rock which consists of volcanic ash is grey in color. It is a remarkable feature that the tuff or tuffite shows the melted appearance like welding. The optical properties of the essential minerals are as follows.
plagioclase; An 45 (core)–60.
augite; (+) 2V=51°. olivine; (−) 2V=85°–88°.

The results of chemical analyses of the rocks described above are given in Table 1 together their halogen contents. Apparently one of the most remarkable feature of these ejectas is their rich content of S and Fe_2O_3. Comparing the results of the chemical analyses with those of the Nakadake lavas[2], all the ejectas have similar characteristics in general, but Al_2O_3, Fe_2O_3, and S contents are rich, while alkalies, especially Na_2O, are poor.

Table 1. Chemical compositions of ejectas on June. 24.

Comp. \ Samp.	Old lava	Andesitic tuff	Coagulated tuff
SiO_2	54.12	54.02	54.27
TiO_2	0.85	0.90	0.86
Al_2O_3	19.23	18.57	18.92
Fe_2O_3	1.79	5.13	6.18
FeO	6.07	3.41	3.46
MnO	0.22	0.17	0.19
MgO	3.96	3.78	3.93
CaO	8.27	8.92	7.02
Na_2O	2.31	2.26	2.13
K_2O	1.40	1.49	1.52
H_2O+	0.86	0.30	0.75
H_2O-	0.03	0.62	0.37
P_2O_5	0.37	0.30	0.26
S	0.29	0.32	0.40
Total	99.77	100.19	100.26
F	0.020	n. d.	0.060
Cl	0.068		0.057

Sublimate Minerals

It has been rare in the activities of the Aso Volcano that many kinds of sublimate minerals are found on the surface or in the cavities of the ejectas like this time.

1) Elements: Sulphur–needle crystal or fine-massive, in druses or on surface of the ejectas.

2) Sulfides: Pyrite–massive or fine-grained on surface of the ejectas with marcasite.
Marcasite–with sulphur or pyrite.

3) Oxides: Magnetite–streak, black, luster metallic and granular aggregates in cavities.

4) Carbonate: Aragonite–colourless, needle crystal in cavities.

5) Sulfates: Anhydrite–small, beautiful crystal in cavities of old lava. Gypsum could not be found in these ejectas, but often found in ash deposits near inactive craters.

6) Silicates: Tridymite–in druses.

7) Halides: Chloromagnesite–colourless or white, platy, deliquescent and easily soluble in water.

Molysite or Lawrencite?–yellow, brownish yellow in color, massive.

8) Hydrous Sulfates: Epsomite–earthy, massive, pale green, $\alpha=1.438$, $\beta=1.462$, $\gamma'=1.468$.

Pickeringite (Ferroan)–conchoidal massive, yellow, $\alpha'=1.477$, $\beta'=1.482$, $\gamma=1.488$.

Melanterite–pale green, surrounding pyrite and marcasite.

Kirovite–with melanterite, pale greenish yellow, $\alpha=1.470$, $\gamma=1.479$, $Z'\wedge C=14°$.

Table 2. Chemical compositions of sublimate minerals.

Comp. \ Samp.	Epsomite	Ferroan pickeringite	Kirovite
Al_2O_3	—	9.83	1.57
Fe_2O_3	—	0.55	—
FeO	0.77	4.26	13.26
MnO	0.32	0.30	0.32
MgO	15.81	2.74	6.97
SO_3	32.37	35.40	30.00
H_2O	50.92	46.76	47.67
Total	100.19	99.84	99.89

The results of chemical analyses of epsomite, pickeringite (ferroan) and kirovite are shown in Table 2.

Volcanic Gas

The writers dared to collect volcanic gasses from small fissures near No. 1 crater after the terrible explosion. (see Fig. 1). The writers went down the crater again about one month later and collected gasses with Dr. Katsura, Mr. Ozawa, and Mr. Kamada. The results of the analyses with radon contents are shown in Table 3. The halogen content as rich and SO_2 and CO_2 are poor during the time of strong activity. Mr. Ozawa found that the composition of later gas is similar to that of Meakan-dake volcano in Hokkido, and Mr. Kamada noticed that the radon content of gas during activity is very poor.

Table 3. The chemical components of gas.

	1 Immediately after the explosion	2 After one month of the explosion
HF	0.43	
HCl	5.61	4.5
H_2S	10.03	7.2
SO_2	7.29	8.7
CO_2	76.32	79.0
R	0.32	0.6
Total	100.00	100.0
Rn*	0.06 mache	2.30 mache

* determined by Kamada

Halogens in Ejectas

The halogens in volcanic gases, volcanic ashes and other ejectas were determined. In general, the halogens in the ejectas show high content during activity. The florine and chlorine contents in volcanic gases, volcanic ashes, tuff, scoria, and old lave are shown in Table 4. This table shows that the atomic ratio of gas agrees with that of the soluble components of ash. Therefore, florine and chlorine in soluble components of ash are considered to be a volcanic gas from magma condensed to ash. The

Table 4. The halogen contents of the ejectas.

	Wt. %		Atomic ratio
	F	Cl	F/Cl×1000
1) gas	0.43	5.61	71
2) Ash	0.82	1.66	97
3) Ash	0.047	0.184	480
4) Tuff	0.060	0.057	845
5) Scoria	0.027	0.063	2186
6) Rock	0.020	0.068	559

1) Showed by vol. %.
2) Wt. % of soluble component of ash (determined by Kamada).
2) 3) 4) 6) The ejectas of the explosion in June 24.
5) The new lava of the eruption in Oct. 30.

quantities of florine and chlorine in gas and soluble components of ash are very large. The atomic ratio of florine and chlorine of ash, scoria, tuff and lavas is larger than those of gas or soluble components of ash. The large quantity of florine in ash and tuff suggests that when gases volatile from magma, the florine may have chemically reacted with the ash which filled the vent. The abundant florine content in ash and tuff shows that the florine reacted chemically with ash or porous tuff which packed the vent. From these data, the halogen contents, paticularly chlorine, increases as the activity progresses in Aso. Accordingly the order of F/Cl ratio shows low value as shown in Table 4. It is a very interesting character and may be used as a guide for forecasting volcanic activity.

Ejecta of the Eruption

During the eruption, No. 1 crater threw up new lava of black, porous molten, scoriaceous blocks, the largest one attaining to about 3 m. in length.

Under the microscope, the phenocrysts of plagioclase, augite, hypersthene, and olivine in the decreasing order are recognized in the porous, yellowish brown, glassy groundmass.

Phenocrysts of plagioclase measure 0.06–5 mm. in length. Zonal structure is common. Brown glass, euhedral tiny crystals of apatite and black magnetite are often included. The composition ranges from An 47 to An 74, and the core always show acidic labradorite or andesine (An 47–50). Augite is the most important among coloured mineral constitue-

Table 5. Chemical composition of new scoria lava on Oct. 29, 1958.

Comp. \ Name.	New scoria
SiO_2	53.79
TiO_2	0.85
Al_2O_3	17.51
Fe_2O_3	2.93
FeO	6.28
MnO	0.12
Mgo	3.84
CaO	8.63
Na_2O	3.83
K_2O	1.48
P_2O_5	0.29
H_2O+	0.91
H_2O-	0.13
Total	100.59
F	0.027
Cl	0.063

Comp. \ Name.		New scoria
Norm	Q.	1.50
	Or.	8.90
	Ab.	32.49
	An.	26.13
	Wo.	6.26
	En.	9.60
	Fs.	8.05
	Mt.	4.18
	Il.	1.52
	Ap.	0.67

nts. It is short prismatic, 0.3–1.5 mm. in length. Twinning on (100) is common. (+) 2V=52°–54°. (+) 2V=50°–51° (small crystals surrounding hypersthene). Hypersthene forms small prismatic crystals, 0.1–0.6 mm. in length and pleochroism is distinct: X=reddish yellow, Y=orangish yellow, Z=yellowish green. (−) 2V=61°, 62°. β=1.697–1.700. Olivine forms more or less rounded crystals.

The chemical composition of this scoria is given in Table 5 with its normative minerals. Comparing with the composition of the Naka-dake lavas(2), the scoria lave is scanty in SiO_2 and rich in CaO and FeO, although the petrologic characters are very similar.

Conclusion

The activity of the Aso volcano in 1958 with a strong explosion in June 24 continued until the beginning of November with an activity of the Strombolian type.

The ejectas of the explosion are all old ejectas which has been packed in the vent, are rich in S and Fe_2O_3, and contain much sublimate minerals. The composition of volcanic gas at this time is very in halogen contents and halogens of ash show the highest value in Japanese volcanoes.

The activity developed the Strombolian type after 3 monthes of the explosion. The ejectas of the eruption are new scoriaceous lava which are olivine hypersthene augite basaltic andesite, and are very similar to the lava of the 1933(3)activity.

References

1. W. R. Foster (1955): Simple method for the determination of the plagioclase feldspars., Am. Min. Vol. 40.
2. H. Matsumoto: Unpublished data.
3. Y. Kawano (1934): The 1932–1933 Activity of the Aso volcano. (In Japanese)., The Jour. of the Jap. Associ. of Min. Petro. and Econ. Geol. Vol. 11, 12.

THE EFFECT OF THE INTERNAL STRESS ON THE LONGITUDINAL MAGNETOSTRICTION IN WEAK MAGNETIC FIELD (V)

Shigeo MATSUMAE

(Received July 17, 1959)

[I] Introduction

The present writer studied the longitudinal magnetostriction of Nickel under various degrees of compression.(1)(2) On further, we study the longitudinal magnetostriction of Nickel under tension. The experimental results are as follows.

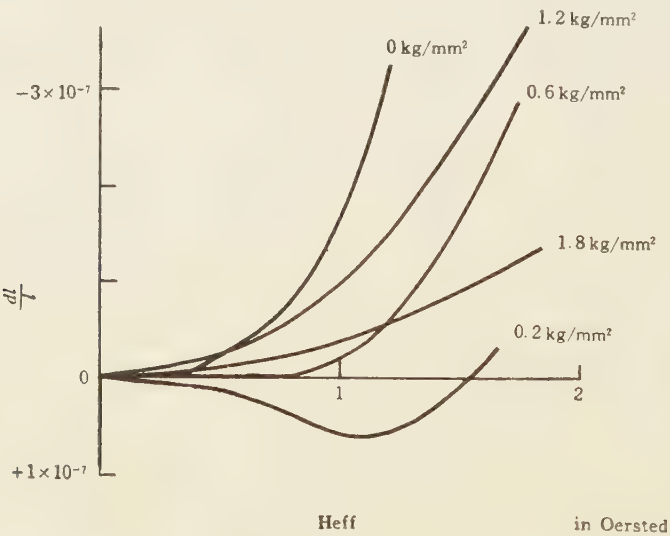
[II] Experimental Results

The experimental method is the same as described in the previous paper.(1)(2)

The specimens are annealed at 900°C for 2 hours and then cooled to the room-temperature with the furnace.

The specimens are, at first, demagnetized with the alternate magnetic field having the maximum amplitude of about 6 Oersted. After demagnetizing, the tensive stress is applied to the specimen and then the longitudinal magnetostriction is measured. The measurement is reported, after the specimen is again demagnetized with the alternate magnetic field having the maximum amplitude of 6 Oersted.

The experimental results are shown in the Fig. 1-A and Fig. 1-B.



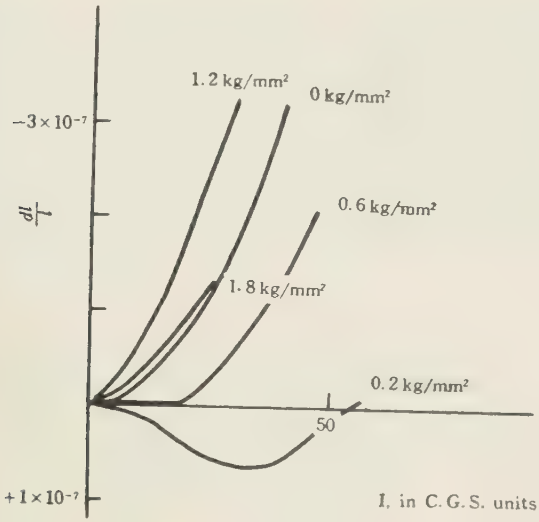


Fig. 1–A The $\frac{dl}{l}$ –I curves and $\frac{dl}{l}$ –Heff curves of nickel under various degrees of tension, when demagnetizing is applied at first and tension subsequently.

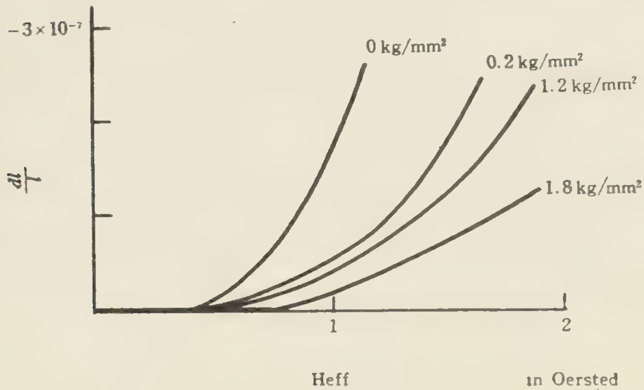


Fig. 1–B The $\frac{dl}{l}$ –Heff curves of nickel under various degrees of tension when demagnetizing and tension is applied at first and then the specimen is again demagnetized with alternate magnetic field of 6 Oersted after application of tension.

As shown in the figure, the longitudinal magnetostriction in weak magnetic field less than about 1 Oersted is negative, under tension of 0.2 kg/mm^2.

In the weak magnetic field, the boundary displacement between the 90-domains occurs. Fig. 2 is a highly schematic representation of the magnetic boundaries. When a weak magnetic field in the direction of the arrow is applied, the boundary-displacements

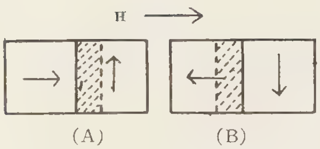


Fig. 2 Domain-interpretation of the effect of a small internal stress on the longitudinal magnetostriction in weak magnetic field.

occurs: in the case of (A), the magnetic domain oriented parallel to the direction of the magnetic field is enlarged by the displacement of the domain boundary at the expense of the magnetic domain oriented at the right angle and, on the other hand, in the case of (B), the reverse is the case. Consequently, the longitudinal magnetostriction does not take place in the weak magnetic field.

But, when a weak magnetic field is applied to the specimen which is subjected to a small external tension in the direction of the magnetizing field, such a wall displacement as is dicated in (B) is easier than that in (A). Consequently, the positive longitudinal magnetostriction will take place in the weak magnetic field. On the contrary, the negative longitudinal magnetostriction takes place, in the case of external compressive stress, as shown in the previous papers.[1][2]

The difference of the absolute value of the longitudinal magnetostriction in weak magnetic field in these two cases is supposed to be caused by the difference of the demagnetizing-factor between in the direction of the axis of the specimen and in the direction of the cross-section of the specimen.

But, when the specimen is demagnetized with the alternate magnetic field or applied to the external tensive stress larger than about 1 kg/mm^2, the longitudinal magnetostriction in weak magnetic field is not negative. This fact indicates that the longitudinal magnetostriction in weak magnetic field is caused by the unstable domain or domain-bundary.

References

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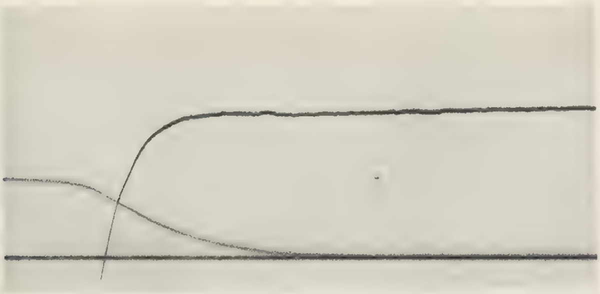
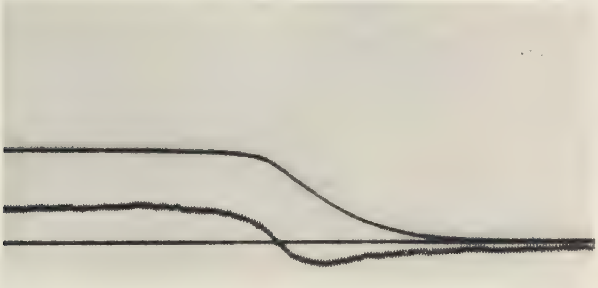
(2) S. Matsumae: The Effect of the Internal Stress on the Longitudinal Magnetostriction in Weak Magnetic Field. Kumamoto Journal of Science. Vol. 4 No. 1 1959.

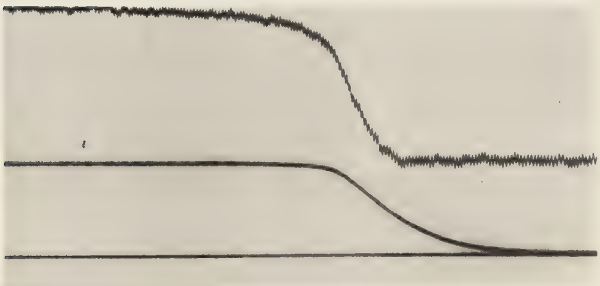


$\sigma = 0 \mathrm{kg/mm^2}$

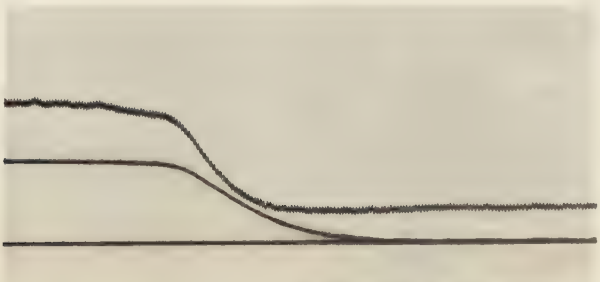
Sensitivity: 1 cm on the oscillograph-paper corresponds to about 2.5×10^{-7}.



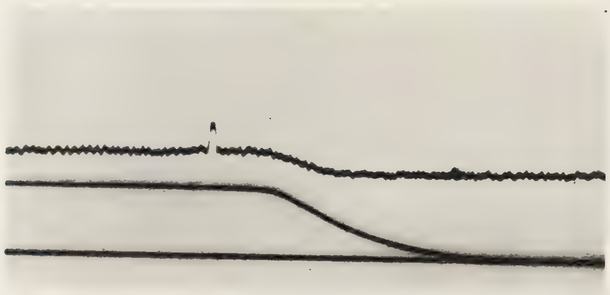
$\sigma = 0.2 \mathrm{kg/mm^2}$



$\sigma = 0.6 \mathrm{kg/mm^2}$

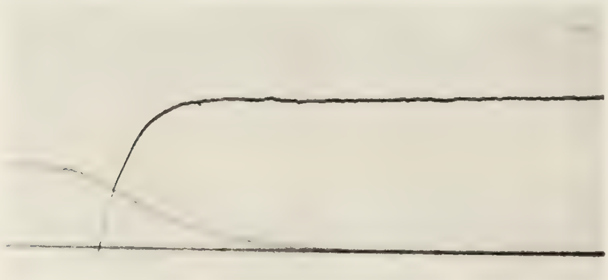


$\sigma = 1.2 \mathrm{kg/mm^2}$



$\sigma = 1.8\ \mathrm{kg/mm^2}$

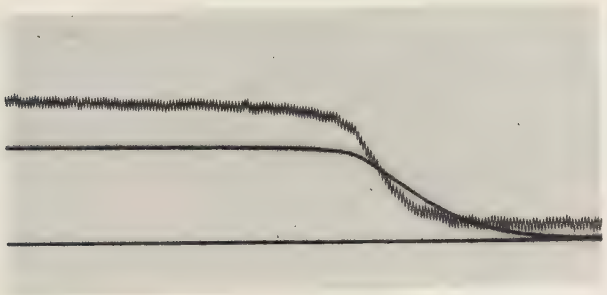
Fig. 1-A The oscillograph-records of the longitudinal magnetostriction of nickel under various degrees of tension, when demagnetizing is applied at first and tension subsequently.
Sensitivity: 1 cm on the oscillograph-paper corresponds to about 2×10^{-7}.



$\sigma = 0\ \mathrm{kg/mm^2}$

Sensitivity: 1 cm on the oscillograph-paper corresponds to about 2.5×10^{-7}.

$\sigma = 0.2\ \mathrm{kg/mm^2}$



$\sigma = 0.6\,\mathrm{kg/mm^2}$

$\sigma = 1.2\,\mathrm{kg/mm^2}$

$\sigma = 1.8\,\mathrm{kg/mm^2}$

Fig. 1–B The change of the oscillograph-records of the longitudinal magnetostriction of nickel under various degrees of tension, when demagnetizing and tension is applied at first and then the specimen is again demagnetized with alternate magnetic field of 6 Oersted.

Sensitivity: 1 cm on the oscillograph-paper corresponds to about 2×10^{-7}.

ERRATA

CORRECTION TO THE PAPER: RELATION BETWEEN A VIBRATION AND ITS GALVANOMETRIC REGISTRATION AND THEIR INITIAL STATES.

Ryuzo ADACHI

(Received July, 15, 1959)

In the paper that appeared in these Journals Vol. 3, Nc. 3 (1958): 212–216, we exchange the initial condition from (6) to

$$\theta_0 = \varphi_0 = \dot{\varphi}_0 = 0, \qquad \dot{\theta}_0 = -K'\dot{x}_0$$

where $K' = 1/L$

and replace following formulae instead of corresponding numbered formulae respectively.

$$\varphi(t) = -KF\left[\sum_{i=1}^{4} \frac{e^{\alpha_i t}}{B_i}\ddot{x}_0 + e^{\alpha_4 t}\int_0^t e^{(\alpha_3-\alpha_4)t}dt \int_0^t e^{(\alpha_2-\alpha_3)t}dt \int_0^t e^{(\alpha_1-\alpha_2)t}dt \int_0^t e^{-\alpha_1 t}\dddot{x}(t)dt\right]$$

$$-K'F\dot{x}_0 \sum_{i=1}^{4} \frac{\alpha_i}{B_i} e^{\alpha_i t} \quad \cdots\cdots\cdots\cdots \quad (10)$$

$$\varphi(t) = KF\sum_{i=1}^{4} \frac{\alpha_i^2}{B_i} e^{\alpha_i t}\left\{x_0 - \alpha_i \int_0^t e^{-\alpha_i t}x(t)dt\right\} + (K-K')F\dot{x}_0 \sum_{i=1}^{4} \frac{\alpha_i}{B_i} e^{\alpha_i t} \quad \cdots\cdots \quad (12)$$

$$\theta(t) = K\left[\sum_{i=1}^{4} \frac{\alpha_i A_i}{B_i} e^{\alpha_i t}\left\{x_0 - \alpha_i \int_0^t e^{-\alpha_i t}x(t)dt\right\} - x(t)\right] + (K-K')\dot{x}_0 \sum_{i=1}^{4} \frac{A_i}{B_i} e^{\alpha_i t} \quad \cdots \quad (13)$$

$$A_i = \alpha_i^2 + 2\varepsilon_1\alpha_i + n_1^2, \quad i = 1, 2, 3, 4 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (14)$$

$$\begin{aligned}
\varphi(t) = F\Big[& \frac{t^2}{2}(-K'\dot{x}_0) + \frac{t^3}{3!}\{-K\ddot{x}_0 + 2(\varepsilon_0+\varepsilon_1)K'\dot{x}_0\} \\
& + \frac{t^4}{4!}\{-K\dddot{x}_0 + 2(\varepsilon_0+\varepsilon_1)K\ddot{x}_0 + (n_0^2+n_1^2-4\varepsilon_0^2-4\varepsilon_0\varepsilon_1-4\varepsilon_1^2-EF)K'\dot{x}_0\} \\
& + \frac{t^5}{5!}\{-K\ddddot{x}_0 + 2(\varepsilon_0+\varepsilon_1)K\dddot{x}_0 + (n_0^2+n_1^2-4\varepsilon_0^2-4\varepsilon_0\varepsilon_1-4\varepsilon_1^2-EF)K\ddot{x}_0 \\
& + (8\varepsilon_0^3+8\varepsilon_0^2\varepsilon_1+8\varepsilon_0\varepsilon_1^2+8\varepsilon_1^3-2\varepsilon_0 n_1^2-2\varepsilon_1 n_0^2-4\varepsilon_0 n_0^2-4\varepsilon_1 n_1^2+4(\varepsilon_0+\varepsilon_1)EF)K'\dot{x}_0\} \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \Big] \quad \cdots\cdots \quad (15)
\end{aligned}$$

$$\theta(t) = K\left[e^{-n_0 t}\left\{n_0^2\int_0^t te^{n_0 t}x(t)dt + (2-n_0 t)n_0\int_0^t e^{n_0 t}x(t)dt + (1-n_0 t)x_0\right\} - x(t)\right]$$

$$+ (K-K')\dot{x}_0 te^{-n_0 t} \quad \cdots\cdots \quad (13')$$

$$\varphi(t)=F\Big[\frac{t^2}{2}(-K'\dot{x}_0)+\frac{t^3}{3!}\{-K\ddot{x}_0+2(n_0+n_1)K'\dot{x}_0\}$$
$$+\frac{t^4}{4!}\{-K\dddot{x}_0+2(n_0+n_1)K\ddot{x}_0-(3n_0^2+4n_0n_1+3n_1^2)K'\dot{x}_0\}+\cdots\cdots\Big]\cdots\cdots\cdots\quad(15)'$$

THE EFFECT OF THE INTERNAL STRESS ON THE LONGITUDINAL MAGNETOSTRICTION IN WEAK MAGNETIC FIELD (VI)

Shigeo MATSUMAE

(Received October 1, 1959)

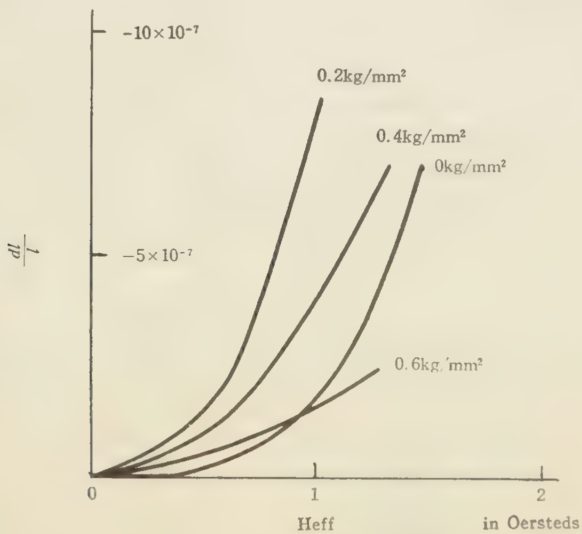
Introduction

We studied the longitudinal magnetostriction of Fe–Ni alloys and Fe–Co alloys in weak magnetic field under various degrees of the external compressive or tensive stress in the elastic limit, according to the negative or positive magnetostriction of the specimen[1][2][3]. In this case, the longitudinal magnetostriction in weak magnetic field is affected greatly by demagnetizing, as described in the previous papers. On further, we describe this phenomena in some details.

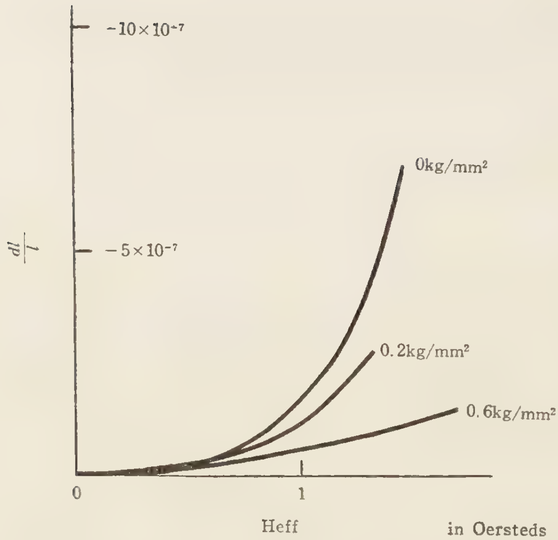
Experimental Results and Discussions

(1) Ni (containning 0.12% Mn)

At first, the longitudinal magnetostriction in weak magnetic field is measured, after the specimen is demagnetized with the alternate magnetic field having the maximum amplitude of 6 Oersted and then applied to the external stress. Secondarily, the measurement of the longitudinal magnetostriction in weak magnetic field is repeated, when the specimen is again demagnetized with the alternate magnetic field having



(Fig. 1–1) before A. C demagnetization



(Fig. 1-2) after A. C demagnetization

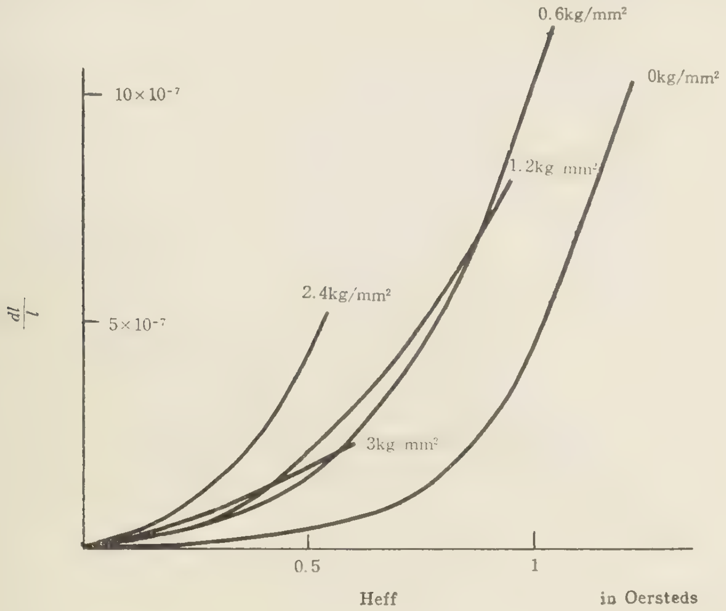
Fig. 1 The $\frac{dl}{l}$ –Heff curves of Ni under various degrees of compression, before and after demagnetizing with the alternate magnetic field of 6 Oersteds.

the maximum amplitude of 6 Oersted after the application of the load. Now, the $\frac{dl}{l}$ –Heff curves in these two cases are shown in the figures.

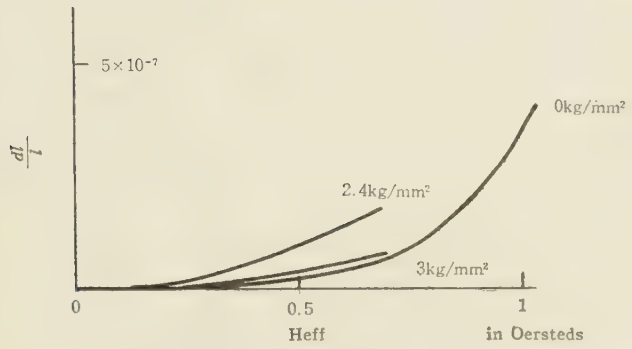
By A. C demagnetization, the longitudinal magnetostriction in weak magnetic field decreases in magnitude and, especially, the longitudinal magnetostriction in the weak magnetic field near H=O decreases so small as to be unobservable. This tendency is generally observed in ferromagnetic metals and alloys.

And the longitudinal magnetostriction in the weak magnetic field bellow about 0.5 Oersteds shows a maximum under the compressive stress of about 0.2 kg/mm², before and after A. C demagnetization. This tendency is also generally observed in ferromagnetic metals and alloys.

(2) Fe–Ni (69.93%) alloy



(Fig. 2–1) before A. C demagnetization

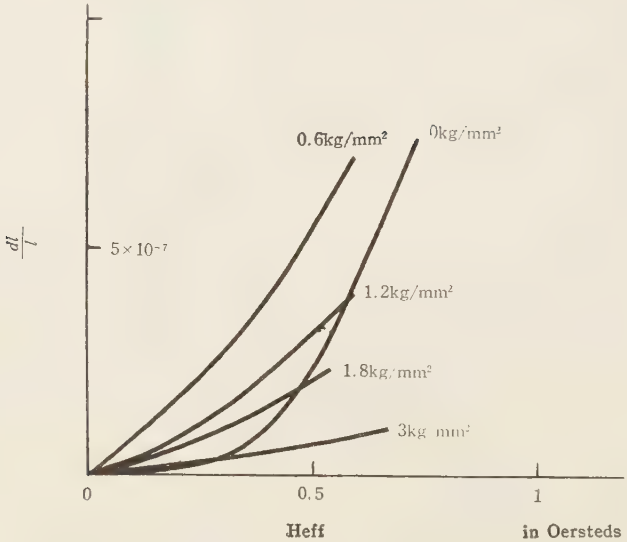


(Fig. 2–2) after A. C demagnetization

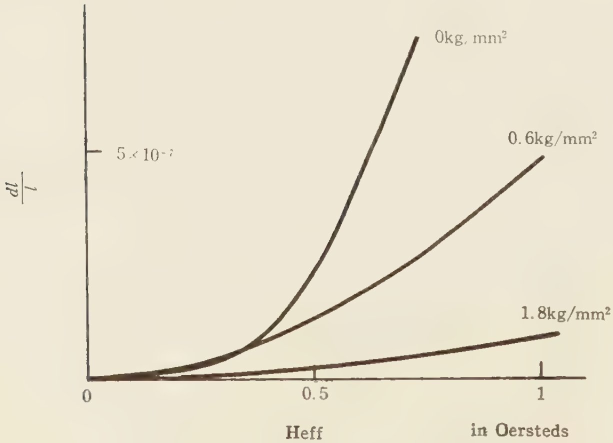
Fig. 2 The $\frac{dl}{l}$ –Heff curves of Fe–Ni (69.93%) alloy under various degrees of tension, before and after demagnetizing with the alternate magnetic field of 6 Oersted.

As shown in the Fig. 2, the longitudinal magnetostriction in the weak magnetic field bellow about 0.6 Oersted shows a maximum under the tension of about 2.4 kg/mm², before and after demagnetizing.

(3) Fe–Ni (50.24%) alloy



(Fig. 3–1) before demagnetization

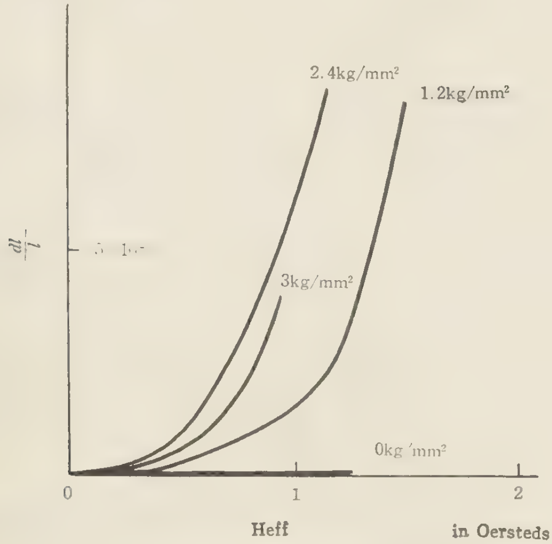


(Fig. 3–2) after demagnetization

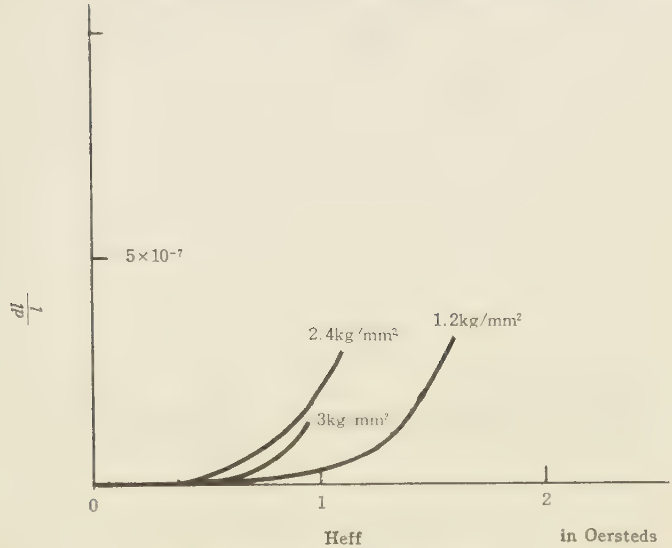
Fig. 3 The $\frac{dl}{l}$ –Heff curves of Fe–Ni (50.24%) alloy under various degrees of tension, before and after demagnetizing with the alternate magnetic field of 6 Oersted.

As shown in the Fig. 3, the longitudinal magnetostriction in the weak magnetic field bellow about 0.3 Oersted shows a maximum under the tension of about 0.6 kg mm², before and after demagnetizing.

(4) Fe– Ni (20.87%) alloy



(Fig. 4–1) before demagnetization

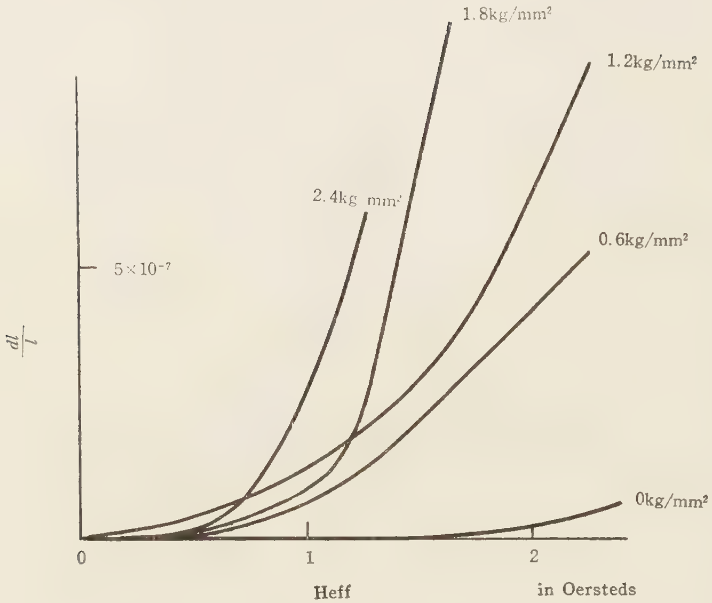


(Fig. 4–2) after demagnetization

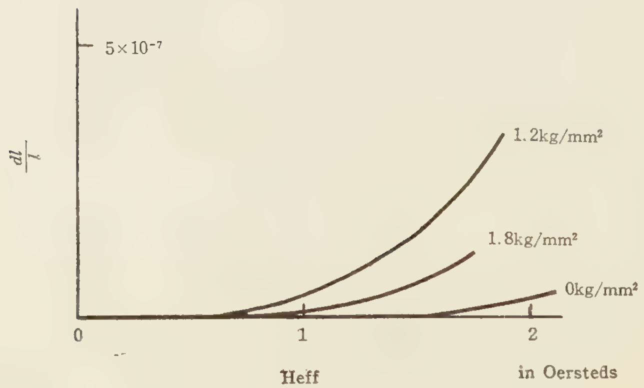
Fig. 4 The $\frac{dl}{l}$ –Heff curves of Fe–Ni (20.87%) alloy under various degrees of tension before and after demagnetizing with the alternate magnetic field of 6 Oersted.

As shown in the Fig. 4, the longitudinal magnetostriction in the weak magnetic field bellow about 1 Oersted shows a maximum under the tension of about 2.4 kg/mm², before and after demagnetizing.

(5) Fe–Co (68.85%) alloy



(Fig. 5-1) before demagnetizing

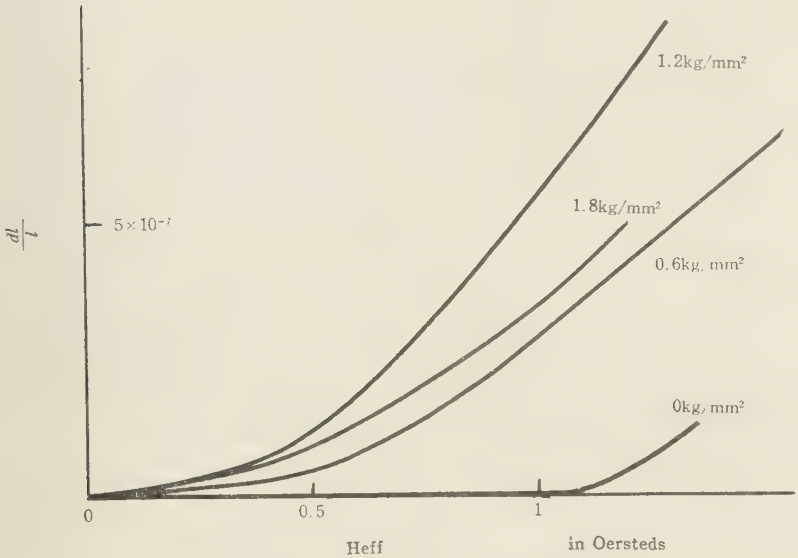


(Fig. 5-2) after demagnetizing

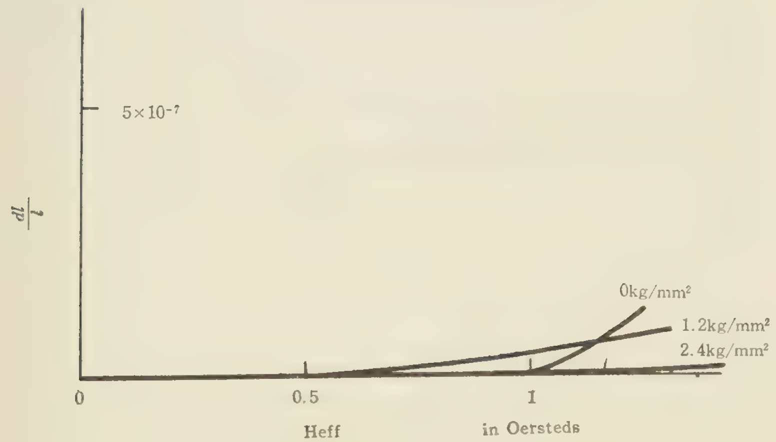
Fig. 5 The $\frac{dl}{l}$ -Heff curves of Fe–Co (68.85%) alloy under various degrees of tension, before and after demagnetizing with the alternate magnetic field of 6 Oersted.

As shown in the Fig. 5, the longitudinal magnetostriction in the weak magnetic field bellow about 1.5 Oersted shows a maximum under the tension of about 1.2 kg/mm², before and after demagnetizing.

(6) Fe-Co (19.59%) alloy



(Fig. 6-1) before demagnetizing

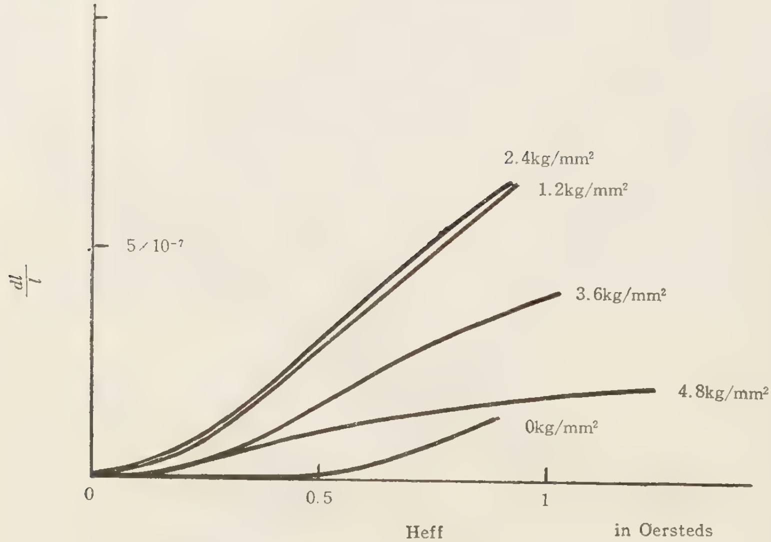


(Fig. 6-2) after demagnetizing

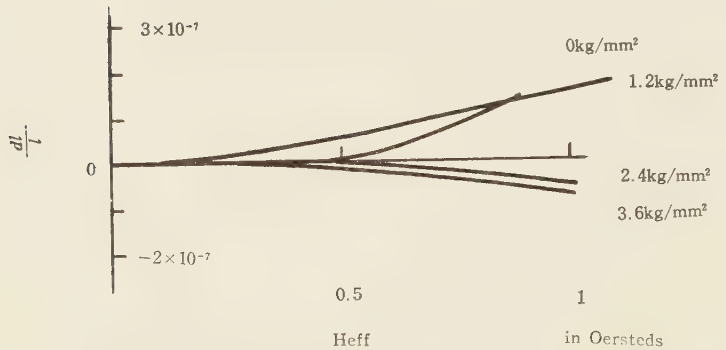
Fig. 6 The $\frac{dl}{l}$ -Heff curves of Fe-Co (19.59%) alloy under various degrees of tension before and after demagnetizing with the alternate magnetic field of 6 Oersted.

As shown in the Fig. 6, the longitudinal magnetostriction in the weak magnetic field bellow about 1 Oersted shows a maximum under the tension of about 1.2 kg/mm², before and after demagnetizing.

(7) Fe



(Fig. 7–1) before demagnetization



(Fig. 7–2) after demagnetization

Fig. 7 The $\frac{dl}{l}$ –Heff curves of Fe under various degrees of tension, before and after demagnetizing with the alternate magnetic field of 6 Oersted.

As shown in the Fig. 7, the longitudinal magnetostriction in the weak magnetic field bellow about 0.5 Oersted shows a maximum under the tension of about 1.2 kg/mm², before and after demagnetizing, when the external tensive stress is smaller than about 2 kg/mm².

As described in the previous paper[1], Fe is a special case.

References

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ON THE GENERALIZED QUANTUM ELECTRODYNAMICS

Seibun SASAKI

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SUMMARY

So far, the invariances of the field theories under the transformations of the parity inversion (P–transformation), the charge conjugation (C–transformation), and the time reversal (T-transformation) are believed to be confirmed experimentally for the electromagnetic interaction as well as for the strong interactions. Therefore, for these interactions, the theories have been developed so as to be kept these invariances hold. But, for the weak interactions, the situation is quite different. It is proved pretty rigidly by various experiments that the CP–invariances do not hold in the decay processes which are liable to the weak interactions.

Usually, the quantum electrodynamics (Q. E. D.) is built up under the assumption that the theory is invariant under both of the proper and imprpper Lorentz transformations. But, as is suggested from the violation of the PC–invariances in the weak interactions, it seems unnecessary to retain the latter invariance or rather, this latter invariance seems to be too strong condition for the theory. If the theory is required to be invariant only under the proper Lorentz transformation, we can not consider the usual Dirac equation as the only one possible form of the linearized equation for the Klein–Gordon equation of motion, and so we must reconsider the properties of Q. E. D. under CPT–transformations from the new point of view.

In this work, we assume that the theory should be invariant only under the proper Lorentz transformation but not under the improper Lorentz transformation. By linearizing the Klein–Gordon equation for the half-spin particle, we get the different from of the Dirac equation from the usual one. This difference does not appear for the free field, but plays a significant role in determining the type of the interaction between various fields. In the case of the interaction with the electromagnetic field, the interaction part of the Hamiltonian is fixed as $ie\,\bar{\psi}\gamma_\mu\psi A_\mu$ if we do not adopt the Pauli-type interaction. The presence of the Pauli-type interaction in the interaction Hamiltonian causes the violence of the PT–invariance even though the degree of the violence is very small. The question whether there is a Pauli-type interaction or not should be determined by such a experiment that detects the validity of the PT–invariances in the electromagnetic phenomena. The connection of our theory with the two-component-theory of the neutrino and with the invariance under the gauge transformation are also discussed. The selection rule about the Yukawa-type interaction between the elementary particles is proposed from our stand-point of view at the last part of this paper.

I. Introduction

As is well known, though the weak interaction may not be invariant under the transformation of the parity inversion (P-transformation) and the transformation of the charge conjugation (C-transformation)…… they are considered to be invariant under the transformation of the time reversal (T-transformation)……, the strong interactions including the electromagnetic interaction are so far believed to be invariant under each of P-, C- and T-transformations. Lately, S. N. Gupta[1] pointed out that the conservation of the parity in the electromagnetic interaction is due to the auxiliary condition for the electromagnetic potential. But there seems to be remained further unsatisfactory points in his discussions.

Firstly, he requires that the electromagnetic interaction is invariant only under the proper Lorentz transformation. The simplest equation for the electromagnetic field in the Heisenberg representation is

$$(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)\,A_\nu=-ie\bar{\psi}\gamma_\mu\psi-ie'\bar{\psi}\gamma_5\gamma_\mu\psi \quad \cdots\cdots (1)$$

where $\bar{\psi}=\psi^*\gamma_4$ and ψ^* is the hermitian conjugate of ψ. Applying the differential operator ∂_μ on both sides of eq. (1), as the left side vanishes identically, one gets

$$\partial_\mu(-ie\bar{\psi}\gamma_\mu\psi-ie'\bar{\psi}\gamma_5\gamma_\mu\psi)=0. \quad \cdots\cdots (2)$$

If one assumes the ordinary Dirac equation for ψ, eq. (2) does not hold unless $e'=0$. If one puts $e'=0$, then eq. (1) reduces to the usual field equation

$$(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)A_\nu=-ie\bar{\psi}\gamma_\mu\psi \quad \cdots\cdots (3)$$

and therefore, the conservability of the parity is assured. Thus, at a first glance, the conservation of the parity in Q. E. D. could be said as is due to the singularity of the field equation for the electromagnetic potential.

Eq. (2) is the continuity equation for the charge-current density and should be derived as one of the results of the Dirac equation. Starting from the ordinary Dirace equation

$$(\gamma_\mu\partial_\mu+\kappa)\psi=0 \quad \cdots\cdots (4)$$

and following the ordinary procedure*)

$$\partial_\mu\rightarrow\partial_\mu-ieA_\mu$$

in the case of the interaction with the electromagnetic field, we find that it is impossible to reconcile the equations (1) and (2) unless 1°) the mass of the Dirac particle is zero or 2°) A_μ is the sum of two parts which have the different parities to each other. In the sense mentioned above, the conclusion of Gupta is a matter of course, or rather it seems to be unsatisfactory. As the ordinary Dirac equation is invariant even under the improper Lorentz transformation, if we take our stand-point

*) Throughout this paper, we adopt the natural unit in which $\hbar=c=1$.

that only the proper Lorentz transformation is reserved, we must seek for the generalized form of the Dirac equation and discuss the properties under the PT–transformations using the latter equation. Such a generalized equation does actually exist[2].

In this paper, we construct the theory of the generalized Q. E. D. by the following procedures:

1°) seeking for the most general form of the equation of such kinds as mentioned above by generalizing the Klein–Gordon equation for the Fermi-particle provided that the wave function of the particle be a four-component spinor,

2°) constructing the generalized Q. E. D. using the equation obtained as above which contains the PT–non-invariant terms in it, and

3°) determining the parameters in the equation obtained by the process 1°) so as to reconcile the equation with the continuity equation of the charge and current density.

Constructing Q. E. D. by this method, if we consider only the usual vector type interaction,it is found that, in spite of the apparent existence of the PT–non-invariant terms, the resulted Q. E. D. becomes equivalent to the ordinary one by applying a suitable transformation. If we introduce, however, the Pauli-like interaction to this generalized Dirac equation, we are able to get Q. E. D. which is invariant under the C–transformation but is no longer invariant under each of the PT–transformations. The proof of the invariance of the theory under the C–transformation will appear in the Appendix.

Now, if we accept *the experimental evidences informing us the invariances of Q. E. D. under each of the PCT–transformations* as undoubtful, we obtain our principle for excluding the Pauli-like interaction from Q. E. D. on the different footing to Gell–Mann's "*principle of the minimal electromagnetic interaction*"[3]. Our principle can be extended to the general case of the Yukawa type interaction between the elementary particles.

On the other hand, if we could determine the coefficient of the Pauli-like interaction term consistent with the experimental evidences so far obtained, and if we could foresee the effects of these PT–non-invariant terms to the experimental results and further, if we could obtain the experimental verifications of these predicted effects, the principle of excluding the Pauli-like interaction mentioned above would become meaningless, that is to say, if it would be the case, the Pauli-like interaction term would no more be ruled out, because there would be no other origin that causes the effects of the violation of the PT–invariance than it.

As for the first possibility to reconcile the equations (1) and (2) previously mentioned, we will discuss it separately, relating to the two-component-theory of the neutrino[4]. Concerning to the second possibility, M. Sachs[5] studied lately the case in which the electromagnetic potential is constructed from the parts of different parities. He dared to change the Maxwell equations and intended to build up the PT–non-invariant Q. E. D.

The last part of this report will be shared to the discussions about the relation of our theory to the gauge transformation and about the selection rule to the form of the Yukawa type interactions.

II. Treatment of the Free Dirac Field

§1. Generalization of the Dirac Equation

According to the general theory, we rewrite the Klein-Gordon equation for the Dirac field

$$(\Box-\kappa^2)\psi=0 \quad \text{where} \quad \Box=\partial_\mu\partial_\mu \qquad (5)$$

into the form

$$d(\partial)\Lambda(\partial)\psi=0, \qquad (6)$$

where

$$\Lambda(\partial)=-\{\Gamma_1\gamma_\mu\partial_\mu+\Lambda_1\kappa\},$$

$$d(\partial)=-\{\Gamma_2\gamma_\mu\partial_\mu-\Lambda_2\kappa\}.$$

Matrices Γ_1, Γ_2, Λ_1 and Λ_2 are determined from the relation $d(\partial)\Lambda(\partial)\equiv(\Box-\kappa^2)$. The conditions for them are

$$\left.\begin{aligned} &\frac{1}{2}(\Gamma_2\gamma_\mu\Gamma_1\gamma_\nu+\Gamma_2\gamma_\nu\Gamma_1\gamma_\mu)=\delta_{\mu\nu}, \\ &\Gamma_2\gamma_\mu\Lambda_1=\Lambda_2\Gamma_1\gamma_\mu, \\ &\Lambda_2\Lambda_1=1. \end{aligned}\right\} \qquad (7)$$

Eliminating Λ_2 and Γ_1 from these relations, we get

$$(\Lambda_1\Gamma_2)\gamma_\mu(\Lambda_1\Gamma_2)\gamma_\nu+(\Lambda_1\Gamma_2)\gamma_\nu(\Lambda_1\Gamma_2)\gamma_\mu=2\delta_{\mu\nu}.$$

Thus, the matrix $\Lambda_1\Gamma_2$ is found to be of the form $e^{\alpha\gamma_5}$ where α is a C-number. For the convenience, we will put it as

$$\Lambda_1\Gamma_2=e^{(A-B)\gamma_5},$$

where A and B are arbitrary C-numbers. Then, from the last two equations of (7) we obtain

$$\left.\begin{aligned} &\Gamma_1=\Gamma_2=e^{A\gamma_5}, \\ &\Lambda_1=e^{-B\gamma_5}, \quad \Lambda_2=e^{B\gamma_5}. \end{aligned}\right\} \qquad (8)$$

Therefore, we get, as the generalized Dirac equation, the relation

$$-\Lambda(\partial)\psi=\{e^{A\gamma_5}\gamma_\mu\partial_\mu+e^{-B\gamma_5}\kappa\}\psi=0 \qquad (9)$$

and as the corresponding operator, the expression

$$d(\partial)=-\{e^{A\gamma_5}\gamma_\mu\partial_\mu-e^{B\gamma_5}\kappa\}. \qquad (10)$$

The ordinary Dirac equation can be obtained by putting A and B zero.

§ 2. Determination of $\bar{\psi}$ and Commutation Relation

The Lagrangian for getting the equation (9) should have the form

$$L=\int\psi^{*}\eta(e^{A\gamma_5}\gamma_\mu\partial_\mu+\kappa e^{-B\gamma_5}\psi)d^4x, \quad \cdots\cdots \quad (11)$$

where η should be so determined that it has the inverse η^{-1} and that the Lagrangian L becoms hermitian.

Then $\bar{\psi}$ is defined as

$$\bar{\psi}=\psi^{*}\eta \quad \cdots\cdots \quad (12)$$

and the commutation relation takes the form

$$\{\psi_\alpha(x),\bar{\psi}_\beta(x')\}_+=id_{\alpha\beta}(\partial)\Delta(x-x'). \quad \cdots\cdots \quad (13)$$

Now, we will proceed to determine the form of $\bar{\psi}$. The relations determining η which come from the hermiticity condition of L are

$$\left.\begin{aligned} &\eta e^{A\gamma_5}\gamma_4=\gamma_4 e^{A^*\gamma_5}\eta^{*}, \\ &-\eta e^{A\gamma_5}\gamma_k=\gamma_k e^{A^*\gamma_5}\eta^{*} \\ &\qquad\qquad (k=1,2,3), \\ &\eta e^{-B\gamma_5}=e^{-B^*\gamma_5}\eta^{*}. \end{aligned}\right\} \quad \cdots\cdots \quad (14)$$

If both A and B are zero, η must be γ_4. Therefore, we will try to find out the form of η by assuming $\eta=S\gamma_4$. Then, the matrix S can be determined from the relations

$$\left.\begin{aligned} &Se^{-A\gamma_5}=e^{-A^*\gamma_5}S^{*}, \\ &S\gamma_4 e^{-B\gamma_5}=e^{-B^*\gamma_5}\gamma_4 S^{*}, \quad S\gamma_k\gamma_4=\gamma_k\gamma_4 S. \end{aligned}\right\} \quad \cdots\cdots \quad (15)$$

From the last of these relations, we can put $S=e^{p\gamma_5}$. C-number constant p is, from the first and the second of eq's (15), found to be

$$p=-\frac{B+B^*}{2}+\frac{A-A^*}{2}=-ReB+iImA,$$

whence

$$\eta=\gamma_4 exp\left(\left\{\frac{B+B^*}{2}-\frac{A-A^*}{2}\right\}\gamma_5\right),$$

and

$$\bar{\psi}=\psi^{+}exp\left(\left\{\frac{B+B^*}{2}-\frac{A-A^*}{2}\right\}\gamma_5\right), \quad \cdots\cdots \quad (16)$$

where

$$\psi^{+}=\psi^{*}\gamma_4.$$

Commutation relation for ψ-field is obtained from (13) by using the expression (16) for

$\bar{\psi}$, the explicit form of which is

$$\{\psi_\alpha(x),\ \bar{\psi}_\beta(x')\}_+ = -i\,\{e^{A\gamma_5}\gamma_\mu\partial_\mu - e^{B\gamma_5}\kappa\}_{\alpha\beta}\,\varDelta\,(x-x'). \quad \cdots\cdots\cdots\cdots\cdots \quad (17)$$

§ 3. Comparison of this Formalism with the Usual one and Some Special Representations

Let us consider the operator of the form $e^{\alpha\gamma_5}$(where α is a C-number) so that the relations

$$\left.\begin{aligned} &e^{\alpha\gamma_5}\gamma_\mu e^{\alpha\gamma_5} = \gamma_\mu, \\ &e^{\alpha\gamma_5}e^{A\gamma_5}e^{-\alpha\gamma_5} = e^{\alpha\gamma_5}e^{-B\gamma_5}e^{\alpha\gamma_5}, \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (18)$$

may hold. Hereafter, we call this kind of operator the "*similarity operator*". The first relation is the identity and the second relation can be satisfied if we choose α as

$$\alpha = \frac{1}{2}(A+B). \quad \cdots\cdots\cdots\cdots \quad (19)$$

Using this operator and introducing the new wave function φ defined by the relation

$$\varphi = e^{-(1/2)(A+B)\gamma_5}\psi, \quad \cdots\cdots\cdots\cdots \quad (20)$$

then the equation of motion (9) reduces to the form

$$(\gamma_\mu\partial_\mu - \kappa)\varphi = 0. \quad \cdots\cdots\cdots\cdots \quad (21)$$

This is just the ordinary Dirac equation. Moreover, this operator transforms the commutation relation (17) into the form

$$\{\varphi_\alpha(x),\ \bar{\varphi}_\beta(x')\}_+ = -i\,(\gamma_\mu\partial_\mu - \kappa)_{\alpha\beta}\varDelta(x-x'), \quad \cdots\cdots\cdots\cdots \quad (22)$$

where

$$\bar{\varphi} = \varphi^* \gamma_4.$$

This is again the ordinary commutation relation for the ordinary Dirac field. Thus, so long as only the free field is concerned, our formalism has no effective difference to the ordinary theory, as it should be.

We can write eq. (6) in the alternative form

$$d(\partial)e^{\beta\gamma_5}\cdot e^{-\beta\gamma_5}\varDelta(\partial)\psi = 0.$$

Here, even if we put newly

$$d'(\partial) = d(\partial)e^{\beta\gamma_5} = -\{e^{(A-B)\gamma_5}\gamma_\mu\partial_\mu - e^{(B+\beta)\gamma_5}\kappa\},$$
$$\varDelta'(\partial) = e^{-\beta\gamma_5}\varDelta(\partial) = -\{e^{(A-B)\gamma_5}\gamma_\mu\partial_\mu + e^{-(B+\beta)\gamma_5}\kappa\},$$

we can develop our theory for the primed system as the same way as before except that the C-number A and B should be replaced by $A-\beta$ and $B+\beta$ respectively. Therefore, by choosing β suitably, we will get the special representation of the theory.

In the following we list up severals of such special representations.

$$
\left.\begin{array}{ll}
1^\circ) & \beta=-A, \\
& (\gamma_\mu\partial_\mu+e^{[illegible]}\kappa)\psi=0, \\
& \bar{\psi}=\psi^+ e^{[illegible]}, \\
& \{\psi_\alpha(x), \bar{\psi}_\beta(x')\}^+=-i(\gamma_\mu\partial_\mu-e^{[illegible]}\kappa)_{\alpha\beta}\Delta(x-x').
\end{array}\right\} \quad \cdots\cdots\cdots\cdots (23)
$$

$$
\left.\begin{array}{ll}
2^\circ) & \beta=-B, \\
& (e^{[illegible]}\gamma_\mu\partial_\mu+\kappa)\psi=0, \\
& \bar{\psi}=\psi^+ e^{(\alpha^*-\alpha)\gamma_5}, \\
& \{\psi_\alpha(x), \bar{\psi}_\beta(x')\}^+=-i(e^{2\alpha\gamma_5}\gamma_\mu\partial_\mu-\kappa)_{\alpha\beta}\Delta(x-x').
\end{array}\right\} \quad \cdots\cdots\cdots\cdots (24)
$$

$$
\left.\begin{array}{ll}
3^\circ) & \beta=\dfrac{A-B}{2}, \\
& (e^{\alpha\gamma_5}\gamma_\mu\partial_\mu+e^{-\alpha\gamma_5}\kappa)\psi=0, \\
& \bar{\psi}=\psi^+ e^{\alpha^*\gamma_5}, \\
& \{\psi_\alpha(x), \bar{\psi}_\beta(x')\}^+=-i(e^{[illegible]}\gamma_\mu\partial_\mu-e^{[illegible]}\kappa)_{\alpha\beta}\Delta(x-x').
\end{array}\right\} \quad \cdots\cdots\cdots\cdots (25)
$$

In these expressions, α stands for $\frac{1}{2}(A+B)$.

In all of these cases, if we put $\varphi=e^{-\alpha\gamma_5}\psi$ and write down the relations for φ, we get the same results as was stated at the beginning of this section.

§ 4. Relation to the Two-Component-Theory of the Neutrino

Let us consider the meaning of the fact that eq. (9) can be transformed into the ordinary form (21) by the transformation (20). This fact can be interpreted as that eq. (9) can be solved in the form of (20). We will examine the situation little more tediously by using the expression (23). The equation of motion for this representation can be written as

$$\{\gamma_\mu\partial_\mu+(a-b\gamma_5)\kappa\}\psi=0, \quad \cdots\cdots\cdots\cdots (26)$$

where

$$a=\cosh 2\alpha, \quad b=\sinh 2\alpha.$$

Here, we used the relation

$$e^{\pm a\gamma_5}=\cosh a\pm\gamma_5\sinh a.$$

Assuming that eq. (26) has the solution of the form

$$\psi\sim(1+k\gamma_5)\varphi \quad \cdots\cdots\cdots\cdots (27)$$

in which φ satisfies the equation $(\gamma_\mu\partial_\mu+\kappa)\varphi=0$, we will determine the condition for k.

Substituting (27) into (26), we get as the condition for k the relation

$$k = \frac{a \pm 1}{b} .$$

i.e. a and b can be expressed with k as follows (together with the relation $a^2 - b^2 = 1$):

$$a = \pm \frac{1+k}{1-k}, \quad b = \pm \frac{2k}{1-k} .$$

Obviouly, $k=1$ is the singular point and we could not take this value. In fact, if we substitute the solution (27) with $k=1$ into eq. (26), we get the relation

$$\{(a-b)\gamma_\mu \partial_\mu + \kappa\}\varphi + \{(a-b)\gamma_\mu \partial_\mu + \kappa\}(\gamma_5 \varphi) = 0.$$

This relation can be satisfied if and only if

$$\varphi = \gamma_5 \varphi \quad \cdots\cdots\cdots\cdots \quad (28)$$

This shows that the wave function φ has only two components, and further, when φ is of two components, the condition that makes the relation $\{(a-b)\gamma_\mu \partial_\mu + \kappa\}\varphi = 0$ valid for the not identically zero wave function φ is

$$E^2 = c^2 p^2,$$

that is to say, $\kappa = 0$ becomes the necessary and sufficient condition for φ being of two components. This shows a good agreement with the two-component-theory of the neutrino.

III. Treatment of the Interaction with the Radiation

§ 5. Vector Type Interaction

The electromagnetic interaction of the vector type is usually introduced into the formalism by the replacement $\partial_\mu \rightarrow \partial_\mu - ieA_\mu$. For the sake of the generalization, however, we will firstly examine the possibility of the replacement

$$\gamma_\mu \partial_\mu \rightarrow \gamma_\mu \partial_\mu - ie\gamma_\mu exp\,(\delta\gamma_5)\, A_\mu \quad \cdots\cdots\cdots\cdots \quad (29)$$

where δ is a C-number. In this section, we do not touch about the problem of the gauge transformation. Concerning this point, we will discuss in the section 7.

The general form of the Lagrangian for the interacting system takes the form

$$\bar{L} = \int \{\psi^+ e^{1/2\{(B+B^*)-(A-A^*)\}\gamma_5} [e^{A\gamma_5}\gamma_\mu(\partial_\mu - ie\ exp(\delta\gamma_5)A_\mu) + e^{-B\gamma_5}\kappa]\psi$$
$$- \frac{1}{4} F_{\mu\nu}F_{\mu\nu}\} d^4x. \quad \cdots\cdots \quad (30)$$

We require that the Lagrangian is hermitic, and find out the condition that the interaction part of the Lagrangian

$$\bar{L}' = -ie \int \psi^+ e^{(\alpha+\alpha^*)\gamma_5} e^{-\delta\gamma_5} \gamma_\mu A_\mu \psi d^4x \quad \cdots\cdots\cdots\cdots \quad (31)$$

$$\left(\alpha = \frac{1}{2}(A+B)\right)$$

is hermitic. This condition will be obtained as the restriction on the value of the C-number δ. It is easily found to be $\delta=\delta^{*}$.

The equations of motion obtained from the Lagrangian (30) lead

$$\{e^{A\gamma_5}\gamma_{\mu}(\partial-ie\ exp(\delta\gamma_5)A_{\mu})+\kappa e^{-B\gamma_5}\}\psi=0, \quad \cdots\cdots \quad (32)$$

$$(\Box\delta_{\mu\nu}-\partial_{\mu}\partial_{\nu})A_{\nu}=-j_{\mu}, \quad \cdots\cdots \quad (33)$$

where

$$j_{\mu}=ie\psi^{+}e^{\alpha\ \alpha\ \gamma_5}\gamma_{\mu}e^{\delta\gamma}\psi. \quad \cdots\cdots \quad (34)$$

From (33), as $\partial_{\mu}(\Box\delta_{\mu\nu}-\partial_{\mu}\partial_{\nu})\equiv 0$, j_{μ} must satisfy the relation (the continuity equation of the charge and current density)

$$\partial_{\mu}j_{\mu}=0.$$

This condition must be consistent with the equation (32). This give the further condition on the value of δ. Calculating the expression $\partial_{\mu}j_{\mu}$ directly from the equation (32), we get

$$\partial_{\mu}j_{\mu}=ie\kappa\psi^{+}[exp(\{(\alpha^{*}-\alpha)+\delta\}\gamma_5)-exp(\{(\alpha^{*}-\alpha)-\delta\}\gamma_5)]\psi.$$

This expression vanishes only when $\delta=0$ because δ is real.

Thus, we must conclude that, even though we give up the invariance of the theory under the improper Lorentz transformation, when there comes the interaction with the electromagnetic field, we must replace ∂_{μ} in the Dirac equation by $\partial_{\mu}-ieA_{\mu}$. If not so, we can not reconcile the Dirac equation with the continuity equation.

As $\gamma_{\mu}(\partial_{\mu}-ieA_{\mu})$ has the same transformation property as $\gamma_{\mu}\partial_{\mu}$ under the similarity transformation, by applying the transformation (20), eq's (32) and (33) reduce to

$$\left.\begin{aligned}&\{\gamma_{\mu}(\partial_{\mu}-ieA_{\mu})+\kappa\}\varphi=0,\\&(\Box\delta_{\mu\nu}-\partial_{\mu}\partial_{\nu})A_{\nu}=-j_{\mu},\end{aligned}\right\} \quad \cdots\cdots \quad (35)$$

where

$$j_{\mu}=ie\varphi^{+}\gamma_{\mu}\varphi.$$

Here we put δ equal to zero.
Thus, as was in the case of the free field, there appears no difference between our formalism and the usual one.

§6. Pauli-Type Interaction

The interaction Lagrangian is

$$\bar{L}'=-\frac{e}{2\kappa}\xi\int\psi^{+}e^{1/2\{(B+B^{*})-(A-A^{*})\}\gamma_5}e^{k\gamma_5}\sigma_{\mu\nu}F_{\mu\nu}\psi d^4x, \quad \cdots\cdots \quad (36)$$

where ξ is a dimensionless parameter and the factor $e^{k\gamma_5}$ is added for the purpose of

getting the general form of the Lagrangian. C-number k should be restricted so as to make the Lagrangian hermitic and to make the continuity equation hold. To make the Lagrangian hermitic, k must satisfy the relation

$$k+k^{*}=-(B+B^{*}) \text{ or } Rek=-ReB.$$

Now, we put

$$k=-\frac{B+B^{*}}{2}+is, \qquad \text{(37)}$$

where s is a real C-number. Then, substituting (37) into (36), we get as the interaction Lagrangian the expression

$$\bar{L}'=-\frac{e}{\kappa}\xi\int\partial_\nu(\psi^{+}e^{-1/2(A-A^{*})\gamma_5}e^{is\gamma_5}\sigma_{\mu\nu}\psi)A_\mu d^4x,$$

which gives us the equations of motion

$$\{e^{A\gamma_5}\gamma_\mu(\partial_\mu-ieA_\mu)+\frac{e}{2\kappa}\xi e^{-\frac{B-B^{*}}{2}\gamma_5}e^{is\gamma_5}\sigma_{\mu\nu}F_{\mu\nu}+\kappa e^{B\gamma_5}\}\psi=0,$$

$$(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)A_\nu=-j_\mu-J_\mu, \qquad \text{(38)}$$

where

$$j_\mu=ie\psi^{+}e^{(\alpha+\alpha^{*})\gamma_5}\gamma_\mu\psi$$

and

$$J_\mu=\frac{e}{\kappa}\xi\,\partial_\nu(\psi^{+}e^{-1/2(A-A^{*})\gamma_5}e^{is\gamma_5}\sigma_{\mu\nu}\psi). \qquad \text{(39)}$$

Obviously

$$\partial_\mu J_\mu=0.$$

Calculating the expression $\partial_\mu j_\mu$ directly from the equation (38), we find that the contributions from the additional Pauli-type interaction term vanishes for all the real values of s.

Transforming ψ into φ by (20) and putting

$$g=s+ImB \ (g\text{: real}),$$

we get the expressions

$$\left.\begin{aligned}\{\gamma_\mu(\partial_\mu-ieA_\mu)+\frac{e}{2k}\xi e^{ig\gamma_5}\sigma_{\mu\nu}F_{\mu\nu}+\kappa\}\varphi=0,\\ j_\mu=ie\varphi^{+}\gamma_\mu\varphi,\\ J_\mu=\frac{e}{\kappa}\xi\,\partial_\nu(\varphi^{+}e^{ig\gamma_5}\sigma_{\mu\nu}\varphi).\end{aligned}\right\} \qquad \text{(40)}$$

Further, if we transform φ into Φ by the relation

$$\varphi = e^{-\frac{i}{2}g\gamma_5}\Phi, \quad \cdots\cdots \quad (41)$$

we get the expressions

$$\left.\begin{aligned} \{\gamma_\mu(\partial_\mu - ieA_\mu) + \frac{e}{2\kappa}\xi\sigma_{\mu\nu}F_{\mu\nu} + \kappa e^{-ig\gamma_5}\}\Phi = 0, \\ j_\mu = ie\Phi^+\gamma_\mu\Phi, \\ J_\mu = \frac{e}{\kappa}\xi\,\partial_\nu(\Phi^+\sigma_{\mu\nu}\Phi). \end{aligned}\right\} \quad \cdots\cdots \quad (42)$$

Thus, we can push the effect of the abandonment of the invariance under the improper Lorentz transformation into one term. The effect of the parity non-conservation (and therefore, the effect of the violence of the conservation under the T-transformation) can be estimated by the magnitudes of the parameters ξ and g.

We cannot get rid of the factor $e^{ig\gamma_5}$ from the formula (40) or (41) ay any kind of transformation unless we put the C-number g equal to zero. If we assume that the violences of the PT-invariances appear from the factor $e^{ig\gamma_5}$ in the above equations, because there are many phenomena in which the Fermi particles play essential roles and that the violences of the PCT-invariances are evident, we can not put g equal to zero from the beginning. The value of g may have some connections to the strange ness of the particles, too. This is one of the future problems.

§ 7. Considerations about the Gauge Invariance

In § 5, we examined the replacement $\gamma_\mu\partial_\mu \rightarrow \gamma_\mu\partial_\mu - ie\,\gamma_\mu\, exp(\delta\gamma_5)A_\mu$. Firstly, we show that this replacement does not keep the theory invariant under the gauge transformation. Corresponding to the transformation

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu\Lambda, \quad \cdots\cdots \quad (43)$$

let us transform ψ as follows:

$$\psi \rightarrow \psi' = e^{ie\Lambda\Gamma}\psi, \quad \cdots\cdots \quad (44)$$

where Γ is an operator containing γ_5.

Substituting (43) and (44) into (32), we get

$$\begin{aligned} \{e^{A\gamma_5}\gamma_\mu e^{-ie\Lambda\Gamma}(\partial_\mu - ie\ exp(\delta\gamma_5)A'_\mu) + \kappa e^{-B\gamma_5}e^{-ie\Lambda\Gamma}\}\psi' \\ + ie\ exp(A\gamma_5)\gamma_\mu\partial_\mu\Lambda\cdot(e^{\delta\gamma_5} - \Gamma)e^{-ie\Lambda\Gamma}\psi' = 0 \end{aligned}$$

In order to make the second term vanish, we must take $\Gamma = e^{\delta\gamma_5}$. Taking this form of Γ, the transformed equation becomes

$$\begin{aligned} e^{ie\Lambda exp(\delta\gamma_5)}\{e^{A\gamma_5}\gamma_\mu(\partial_\mu - ie\ exp(\delta\gamma_5)\cdot A_\mu) \\ + \kappa e^{-B\gamma_5}e^{-2ie\Lambda exp(\delta\gamma_5)}\}\psi' = 0. \end{aligned}$$

This means that the replacement (29) destroys the invariance under the gauge trans-

formation. In order to make the theory gauge invariant, we must take the value of δ equal to zero. This result agrees with the one obtained in the section 5. Thus, from the consideration about the gauge transformation, we get the same result as was obtained before. The condition that the requirement of the invariance under the gauge transformation must be fulfiled for Q.E.D., is found to give the strict restriction to the choice of the form of the interaction Hamiltonian, even in the case in which the invariance under the improper Lorentz transformation is given up and only the invariance under the proper Lorentz transformation is retained.

We have shown in §5 that the interaction of the form $ie\bar{\psi}\gamma_\mu e^{\delta\gamma_5}\psi A_\mu$ can not be reconciled with the continuity equation of the charge and current density. As the continuity equation is derived from the variation principle in wihich the function Λ in the expressions (43) and (44) varies independently, it is very natural that the interaction conflicting with the continuity equation can not be invariant under the gauge transformation. Thus, the requirement of the gauge invariance of the theory plays an essential role for determining the interaction form in the quantum electrodynamics.

IV. Selection Rule for the Yukawa Type Interaction

Usually, when we want to take into into account the effects of the violence of each of the PCT-invariances, we modify only the interaction Hamiltonian so as to explain the experimental results exhibiting the above violence. But, this procedure seems to be rather artificial and there seems to be no clearcut reason to do so. In order to avoid this artificial and undistinguish character of the procedure mentioned above, we must construct the whole theory to be free from the condition of the invariance under the improper Lorentz transformation. Of course, the effects under P-, C-and T-transformations must appear only when there are some interactions between various fields and must vanish in the case of the free fields. That is to say, the mixing of the parity in the interaction Hamiltonian, for example, comes from the abandonment of the invariance under the improper Lorentz transformation. As was mentioned in §5 and §6, for the electromagnetic interaction it is possible to consider the two types of the interactions, the vector type interaction and the Pauli type one. Further, we find that, for the vector type interaction, there holds the P-, C-and T-invariances, but for the Pauli type interaction there does not hold such kinds of invariances any more. Of course, in Q. E. D., such a situation comes chiefly from the invariance of the theory under the gauge transformation. In the other interactions concerning such as the mesons or the hyperons, it is unnecessary to consider the gauge invariance, and therefore, the situations are more or less different from the Q. E. D. Nevertheless, we will note another feature of the Q. E. D. The vector type interaction and the Pauli-type (tensor type) interaction have the different properties for the similarity transformation. The analogous fact can quite generally be said for the other types of the interactions. We can classify the Yukawa type interactions into two categories by the properties under the similarity transformation. One of them has the same property as $\psi^+\gamma_\mu\psi$ and the other has the same property as $\psi^+\sigma_{\mu\nu}\psi$. Let us call them the first group and the second group, respectively. The members of

each group are as follows:

1_{st} *group:* $\psi^{+}\gamma_{\mu}\psi$, $\psi^{+}\gamma_{5}\gamma_{\mu}\psi$,

2_{nd} *group:* $\psi^{+}\psi$, $\psi^{+}\gamma_{5}\psi$, $\psi^{+}\gamma_{\mu}\gamma_{\nu}\psi$, $\psi^{+}\gamma_{5}\gamma_{\mu}\gamma_{\nu}\psi$.

These two groups are characterized by the numbers of the matrix γ_{μ}, which means the difference under the similarity transformation.

When the Bose particles interact with the Fermi particles through the Yukawa type interactions, they can take the two types of the coupling forms. One of them is the direct interaction and the other the derivative one. They belong to the different groups owing to the difference of the number of the matrix γ_{μ}. We utilize these properties for setting up the selection rule for the Yukawa type interaction.

We assume that the first group of the interaction is obtained from the free field equation by the replacement*)

$$\gamma_{\mu}\partial_{\mu}\rightarrow\gamma_{\mu}\partial_{\mu}-igU^{\circ}_{\mu},$$

where U°_{μ} stands for $\partial_{\mu}U(\gamma_{5}\partial_{\mu}U)$ for the scalar (pseudoscalar) Bosons and $U_{\mu}(\gamma_{5}U_{\mu})$ for the vector (pseudovector) Bosons. Then, as in Q. E. D., $\gamma_{\mu}(\partial_{\mu}-ieU^{\circ}_{\mu})$ shows the same property as $\gamma_{\mu}\partial_{\mu}$ under the similarty transformation, and so, again as in Q.E.D., it shows no PUT–non-invariant effect. While, if we introduce the second group interaction in the similar way as in Q. E. D., it shows the very PCT–non-invariant effects. Therefore, it seems reasonable to assume that the interaction which shows the PCT–non-invariant characters is of the second group, and that the interactions belonging to the first group do not show the PCT–non-invariant effect.

Thus we conclude that if the electromagnetic interaction as well as the strong interactions are invariant under each of the PCT–transformations, they are all belonging to the first group. Concerning these points, we will discuss in detail in the forthcoming paper, relating to the invariance properties under the proposed gauge-like transformations.

By considering the parity doublet theory for the Boson, the author already reached the same results(5). Thus, starting from the different assumptions, we can get the same conclusions about the types of the Yukawa interactions.

V. Discussions and Conclusions

The problem whether there is a Pauli-type interaction or not is tedious and has been discussed for a long time. Of course, the final decision would rely upon the experiment. If there were, they would be recognized by the discovery of the electromagnetic phenomena showing the violence of the invariances under each of the PT–transformations.

The contribution of the Pauli type interaction on the anomalous magnetic moment of the Fermi particle is estimated by iterating the equation (42) and is found

*) In this place, we consider the neutral field. The case of the charged field is obtained easily by the same manner.

to be $e\xi\cos\vartheta\cdot\sigma_{\mu\nu}F_{\mu\nu}$. If we take the value of $|\xi\cos\vartheta|$ about the order of α^3 (α is a fine structure constant), the presence of the Pauli type interaction does not conflict with the customary experiments. Of course, in estimating the order of $|\xi\cos\vartheta|$, we must consider the occurrence of the static electric dipole moment which appears when the PT-invariance destroys, or the energy levels of the hydrogen atom including the Lamb shift. But, as the general tendencies, there seems to be several improper properties especially in the renormalization theory. If we can exclude it reasonably, it must be very happy. Thus, the experiment for detecting the non-invariance under the P-transformation becomes very important.

Assuming only the invariance under the Lorentz transformation, our conclusion can be listed up as follows.

1°) *The Dirac equation can be generalized, but the effect of the generalization does not appear for the case of the free field.*

2°) *This effect does not appear for the vector type electromagnetic interaction, too.*

3°) *The Pauli type interaction exhibits the non-invariance under each of the PT-transformations. The violence of the invariance is not introduced artificially, but is the naturally result of the theory.*

4°) *The vector type electromagnetic interaction is obtained if and only if we replace* ∂_μ *by* $\partial_\mu - ieA_\mu$ *assuming that* ψ *is of four components.*

5°) *The two-component-theory of the neutrino is smoothly obtained as the special case of the generalized Dirac equation.*

6°) *In general, the Yukawa type interactions are classified into two groups, the first of which is obtained by the replacement* $\gamma_\mu\partial_\mu \rightarrow \gamma_\mu(\partial_\mu - igU^\circ_\mu)$ *and does not show the PT-non-invariances. The strong interactions as well as the electromagnetic interaction both of which are belived to be invariant under each of the PCT-transformations belong to this group.*

7°) *The second group shows the PT-non-invariant effects. The weak interactions may also belong to this group. As is in the Pauli type interaction, the violences of the invariances are the natural result of the theory.*

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Appendix. Properties of (42) under the P–, C–and T–transformations

1°) C–transformation

The equation satisfied by $\bar{\Phi} = \Phi^* \gamma_4$ is obtained from (42) as

$$\bar{\Phi}(\gamma_\mu(\overleftarrow{\partial}_\mu + ieA_\mu) - \frac{e}{2\kappa}\xi\sigma_{\mu\nu}F_{\mu\nu} - \kappa e^{-ig\gamma_5}) = 0,$$

where $\overleftarrow{\partial}_\mu$ means that the operation ∂_μ should be applied to $\bar{\Phi}$. This equation can be rewritten as

$$\{\gamma_\mu^T(\partial_\mu + ieA_\mu) - \frac{e}{2\kappa}\xi\sigma_{\mu\nu}^T F_{\mu\nu} - \kappa e^{-ig\gamma_5}\}\bar{\Phi}^T = 0,$$

where γ_μ^T, $\sigma_{\mu\nu}^T$ etc. mean the transposed matrices of the γ_μ, $\sigma_{\mu\nu}$ etc. Let us transform $\bar{\Phi}^T$ into Φ' by the relation

$$\Phi' = C\bar{\Phi}^T,$$

where C is the matrix having the following propecties:

$$C^T = -C, \qquad C\gamma_\mu^T C^{-1} = -\gamma_\mu,$$
$$C\gamma_5^T C^{-1} = \gamma_5, \qquad C\sigma_{\mu\nu}^T C = \sigma_{\mu\nu}.$$

Then as the equation satisfied by Φ', we get

$$\{\gamma_\mu(\partial_\mu + ieA_\mu) - \frac{e}{2\kappa}\xi\sigma_{\mu\nu}F_{\mu\nu} + \kappa e^{-ig\gamma_5}\}\Phi' = 0.$$

The difference of this equation with (42) is only the sign of the charge, and therefore, if we adopt the Heisenberg's symmetrization procedure for the Lagrangian, we get the C–invariant formalism; that is to say, it is enough to replace j_μ and J_μ by

$$j_\mu = \frac{1}{2} ie[\bar{\Phi}, \gamma_\mu \Phi],$$

$$J_\mu = \frac{e}{2\kappa}\xi\partial_\nu[\bar{\Phi}, \sigma_{\mu\nu}\Phi].$$

2°) P–transformation

By the space reflection $x_k \to x_k' = -x_k$, $x_4 \to x_4' = x_4$, the quantities of the electromagnetic field are transformed as

$$\vec{A} \to -\vec{A}, \; A_4 \to A_4; \; \vec{E} \to -\vec{E}, \; \vec{H} \to \vec{H}.$$

Introducing the matrix S having the properties

$$S^{-1}\gamma_k S = -\gamma_k (k = 1.2.3), \; S^{-1}\gamma_4 S = \gamma_4$$

we transform Φ into Φ' by the relation $\Phi = S\Phi'$. Then we get

$$\{\gamma_\mu(\partial_\mu - ieA_\mu) + \frac{e}{2\kappa}\xi\sigma_{\mu\nu}F_{\mu\nu} + \kappa e^{i\vartheta\gamma_5}\}\Phi' = 0.$$

This shows that the formalism is not invariant under the P-transformation.

3°) *T-transformation*

In this case, $x_k \to x'_k = x_k$, $x_4 \to x'_4 = -x_4$ and $\vec{E} \to \vec{E}$, $\vec{H} \to -\vec{H}'$ By transforming Φ into Φ' by the matrix S having the properties

$$S^{-1}\gamma_k S = \gamma_k, \quad S^{-1}\gamma_4 S = -\gamma_4$$

we get as the equation for Φ'

$$\{\gamma_\mu(\partial_\mu - ieA_\mu) + \frac{e}{2\kappa}\xi\sigma_{\mu\nu}F_{\mu\nu} + \kappa e^{i\vartheta\gamma_5}\}\Phi' = 0$$

which shows the violence of the T-invariance of the formalism.

THE MOTION OF A PENDULUM WHOSE SUPPORTING AXIS MOVES TWO DIMENSIONALLY

Ryuzo ADACHI

(Received October 26, 1959)

§1. Introduction

Ordinarily we use so-called vibroscopes to observe the vibrations of various bodies (e. g., earthquake, building and bridge etc.). There are many kinds of vibroscopes, most of them are consisted of a pendulum. For example, to know the horizontal component of a vibration of some body, we can use a vertical pendulum (ordinary pendulum); that is, by driving its supporting axis by the vibration which is to be observed, we can determine the original vibration from the record of the vibration of the pendulum. In this case, the relation between the record of the vibration of the pendulum and the original vibration is clearly known when the vertical component is zero. But in this case, if the vertical component is not zero, what will become the relation? It seems that there is no dissertation treating this problem *generally*, and only a special case is discussed by Dr. Y. SUEHIRO.

In this paper, the author discussed this problem generally, and obtained the solution of the motion of a pendulum provided that the two components of the motion of the supporting axis of the pendulum be known, and also some special cases and an example were given.

§2. Fundamental formulas

There are many types of vibroscopes consisted of a pendulum. For example, in the seismology, there are vertical pendulum, horizontal pendulum and inverted pendulum etc. (Fig. 1: a, b, c) to record one of the horizontal component of the motion of the earth, and some pendulums whose weights are supported by springs (Fig. 1: d, e) to record the vertical component of the motion of the earth. But the differential equations expressing their motions are almost the same and, as an example, we will discuss the motion of a pendulum which is shown by (d) in the Fig. 1.

As shown in the Fig. 2, we take the x-axis downwards along the vertical line, the y-axis along a horizontal line, and let a pendulum whose center of gravity is $G(\xi, \eta)$ oscillate around the axis passing C and perpendicular to the xy-plane. Let θ be the displacement angle of the pendulum at any instance, then assuming that $|\theta|$ is small, we have

$$a_0 f_0 = mgl, \qquad f = f_0 + \alpha\theta \qquad (1)$$

$$m\ddot{\xi} = f_x + mg - f, \qquad m\ddot{\eta} = f_y \qquad (2)$$

$$I_G\ddot{\theta}=f_y l\theta-f_x l+(l-a_0)f \qquad \cdots\cdots 3$$

$$\xi=x+l\theta, \qquad \eta=y+l \qquad \cdots\cdots 4 \ .$$

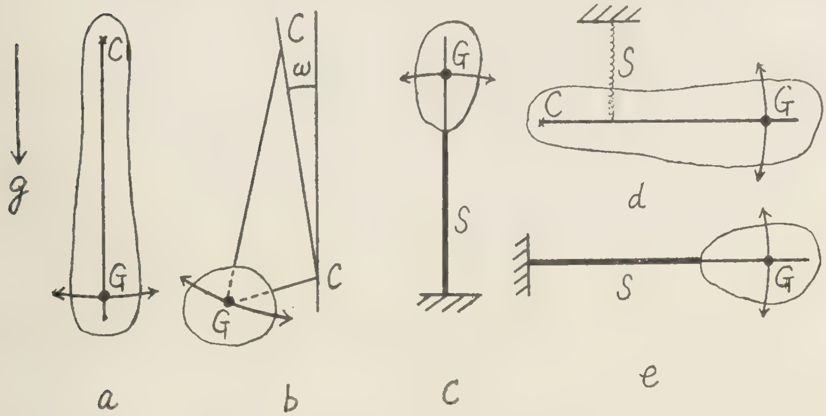

Fig. 1 Some types of pendulums.
C–supporting axis, **S**–spring, **G**–center of gravity
a, b, c ··· for horizontal motion
d, e ··· for vertical motion

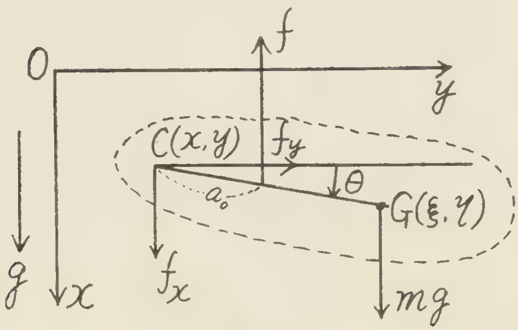

Fig. 2

where

$l=CG$
$m=$the mass of the pendulum
$I_G=$the moment of inertia of the pendulum with respect to the axis passing G and parallel to the axis of the pendulum
$f_x=$the x-component of the resisting force acting on the axis (C) of the pendulum ······ 5 .
$f_y=$the y-component of the resisting force acting on the axis (C) of the pendulum
g $=$the acceleration of gravity

α = some constant concerning S
f = the tension of S
f_0 = the tension of S when $\overline{CG}//y$-axis and the pendulum is at rest in the state of equilibrium

We can easily derive following formula from above formulas,

$$I\ddot{\theta}+a_0\alpha\theta=ml(\ddot{y}\theta-\ddot{x}) \quad \cdots\cdots (6)$$

where $I=I_G+ml^2$.

Moreover, if a damping term $k\dot{\theta}$ acts on the motion of this pendulum, (6) becomes

$$\ddot{\theta}+2a\dot{\theta}+(b^2-K\ddot{y})\theta=-K\ddot{x} \quad \cdots\cdots (7)$$

where

$$2a=k/I>0,\ \ l_0=I/ml,\ \ b^2=a_0\alpha/I,\ \ K=1/l_0.$$

Putting

$$L[\theta]=\ddot{\theta}+2a\dot{\theta}+b^2\theta \quad \cdots\cdots (8)$$

gives

$$L[\theta]=-K\ddot{x}+K\ddot{y}\theta \quad \cdots\cdots (9).$$

Next, as the initial conditions, we take $\theta=0$, $\dot{\xi}=0$ at $t=0$, and get

$$\theta(0)=0,\ \ \dot{\theta}(0)=-K'\dot{x}(0) \quad \cdots\cdots (10)$$

where $K'=1/l$.

As mentioned above, the differential equations for the motions of other kinds of pendulums are the same types as (9), therefore, in the following section, we consider the solution of (9) under the conditions (10) provided that $x(t)$, $y(t)$ be known.

§ 3. The solution of (9) satisfying the conditions (10)

Let $\varphi_0(t)$ be the solution of

$$L[\varphi_0]=-K\ddot{x} \quad \cdots\cdots (11)$$

satisfying the conditions

$$\varphi_0(0)=0,\ \ \dot{\varphi}_0(0)=-K'\dot{x}(0) \quad \cdots\cdots (12),$$

and put

$$\theta(t)=\varphi_0(t)+\varphi(t) \quad \cdots\cdots (13).$$

Then substituting (13) into (9) gives

$$L[\varphi]=K\ddot{y}\varphi_0(t)+K\ddot{y}\varphi(t) \quad \cdots\cdots (14),$$

and

$$\varphi(0)=0, \quad \dot{\varphi}(0)=0 \quad \cdots\cdots \quad (15)$$

from (10), (12) and (13).
Therefore if we can find $\varphi_0(t)$ and $\varphi(t)$ from (11), (12) and (14), (15) respectively, $\theta(t)$ defined by (13) is our required solution.

At first, (11) is a linear differential equation of the second order with respect to $\varphi_0(t)$ with constant coefficients, and we can solve it by the well known method; that is, by putting

$$\alpha_1=-a+\sqrt{a^2-b^2}, \quad \alpha_2=-a-\sqrt{a^2-b^2} \quad \cdots\cdots \quad (16)$$

$$N(s,t)=\frac{1}{\alpha_1-\alpha_2}\{e^{\alpha_1(t-s)}-e^{\alpha_2(t-s)}\} \quad \cdots\cdots \quad (17)$$

we get the solution

$$\varphi_0(t)=\frac{K'\dot{x}(0)}{\alpha_1-\alpha_2}(e^{\alpha_2 t}-e^{\alpha_1 t})-K\int_0^t N(s,t)\ddot{x}(s)ds \quad \cdots\cdots \quad (18).$$

This formula can be transformed as follows

$$\varphi_0(t)=-Kx(t)+\frac{Kx(0)}{\alpha_1-\alpha_2}(\alpha_1 e^{\alpha_1 t}-\alpha_2 e^{\alpha_2 t})+\frac{(K-K')\dot{x}(0)}{\alpha_1-\alpha_2}(e^{\alpha_1 t}-e^{\alpha_2 t})$$
$$+\frac{K}{\alpha_1-\alpha_2}\{\alpha_2^2 e^{\alpha_2 t}\int_0^t e^{-\alpha_2 s}x(s)ds-\alpha_1^2 e^{\alpha_1 t}\int_0^t e^{-\alpha_1 s}x(s)ds\} \quad \cdots\cdots \quad (19).$$

Here we assumed that $\alpha_1 \neq \alpha_2$, that is, $a \neq b$. In the case of $a=b$, taking the limits of above formulas, we get following expressions

$$N(s,t)=(t-s)e^{-a(t-s)} \quad \cdots\cdots \quad (17)'$$

$$\varphi_0(t)=-K'\dot{x}(0)te^{-at}-K\int_0^t N(s,t)\ddot{x}(s)ds \quad \cdots\cdots \quad (18)'$$

or

$$\varphi_0(t)=-Kx(t)+Kx(0)(1-at)e^{-a}+(K-K')\dot{x}(0)te^{-at}$$
$$+Kae^{-at}\{(2-at)\int_0^t e^{as}x(s)ds+a\int_0^t e^{as}sx(s)ds\} \quad \cdots\cdots \quad (19)'.$$

Next, putting

$$\left.\begin{array}{l} L[\varphi_j(t)]=K\ddot{y}(t)\varphi_{j-1}(t) \\ \varphi_j(0)=\dot{\varphi}_j(0)=0, \quad j=1,2,3,\cdots\cdots \end{array}\right\} \quad \cdots\cdots \quad (20)$$

gives

$$\varphi_j(t)=\int_0^t N(s,t)\{K\ddot{y}(s)\varphi_{j-1}(s)\}ds \quad \cdots\cdots \quad (21)$$

and we can find $\varphi_1(t)$, $\varphi_2(t)$. ········· successively by this formula. Now, $t-s\geqq 0$ when $0\leqq s\leqq t$ and the real parts of α_1, α_2 are not positive no matter when α_1, α_2 may be real or imaginary, hence $|N(s,t)|$ has an upper bound. Therefore if $|\ddot{x}(t)|$, $|\ddot{y}(t)|$ are bounded uniformly, there exist some constants G_N, G_x, G_y, G_0 such that

$$\left.\begin{array}{ll} |N(s,t)|<G_N, & |\ddot{x}(t)|<G_x \\ |\ddot{y}(t)|<G_y, & |\varphi_0(t)|<G_0 \end{array}\right\} \quad \cdots\cdots\cdots\cdots \quad (22)$$

independently of t which is finite, and we get following inequalities

$$|\varphi_1(t)|\leqq K\int_0^t|N(s,t)\ddot{y}(s)\varphi_0(s)|ds<KG_NG_yG_0t=G_0Gt$$

where $G=KG_NG_y$

$$|\varphi_2(t)|<G\int_0^t|\varphi_1(s)|ds<G_0G^2\int_0^t sds=G_0G^2\frac{t^2}{2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

generally

$$|\varphi_j(t)|<G_0G^j\frac{t^j}{j!} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (23).$$

Next, using the relations

$$N(t,t)\equiv 0,\quad \left[\frac{\partial}{\partial t}N(s,t)\right]_{s=t}\equiv 1$$

we get from (21)

$$\dot{\varphi}_j(t)=K\int_0^t\frac{\partial}{\partial t}N(s,t)\ddot{y}(s)\varphi_{j-1}(s)ds$$

$$\ddot{\varphi}_j(t)=K\ddot{y}(t)\varphi_{j-1}(t)+K\int_0^t M(s,t)\ddot{y}(s)\varphi_{j-1}(s)ds \quad \cdots\cdots\cdots\cdots \quad (24),$$

where

$$\left.\begin{array}{l} M(s,t)=\dfrac{\partial^2}{\partial t^2}N(s,t)=\dfrac{1}{\alpha_1-\alpha_2}\{\alpha_1^2e^{\alpha_1(t-s)}-\alpha_2^2e^{\alpha_2(t-s)}\} \\ \qquad\qquad \text{or} \qquad =\{a^2(t-s)-2a\}e^{-a(t-s)} \end{array}\right\} \quad \cdots\cdots\cdots \quad (25).$$

It is obvious that $|M(s,t)|$ has an upper bound independently of t which is finite, and there exists a constant G_M such that

$$|M(s,t)|<G_M \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (26),$$

hence we get

$$|\ddot{\varphi}_j(t)|<KG_y|\varphi_{j-1}(t)|+KG_yG_M\int_0^t|\varphi_{j-1}(s)|ds$$

from (24), and substituting (23) gives

$$|\ddot{\varphi}_j(t)| < KG_yG_0G^{j-1}\frac{t^{j-1}}{(j-1)!} + KG_MG_yG_0G^{j-1}\frac{t^j}{j!}$$
$$= KG_0G_y\left\{\frac{(Gt)^{j-1}}{(j-1)!} + \frac{G_M}{G}\frac{(Gt)^j}{j!}\right\}.$$

Finally we have

$$\sum_{j=1}^{\infty}|\ddot{\varphi}_j(t)| < KG_0G_y\sum_{j=1}^{\infty}\left\{\frac{(Gt)^{j-1}}{(j-1)!} + \frac{G_M}{G}\frac{(Gt)^j}{j!}\right\}$$
$$= KG_0G_y\{e^{Gt} + \frac{G_M}{G}(e^{Gt}-1)\} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (27).$$

Therefore $\sum_{j=1}^{\infty}\ddot{\varphi}_j(t)$ is uniformly convergent in any finite interval of t, and then $\sum_{j=1}^{\infty}\dot{\varphi}_j(t)$ and $\sum_{j=1}^{\infty}\varphi_j(t)$ are also uniformly convergent in the same interval of t. Accordingly

$$L\left[\sum_{j=1}^{\infty}\varphi_j(t)\right] = \sum_{j=1}^{\infty}L[\varphi_j(t)] = K\ddot{y}(t)\sum_{j=0}^{\infty}\varphi_j(t)$$
$$= K\ddot{y}(t)\varphi_0(t) + K\ddot{y}(t)\sum_{j=1}^{\infty}\varphi_j(t),$$

and moreover

$$\sum_{j=1}^{\infty}\varphi_j(0) = 0, \quad \sum_{j=1}^{\infty}\dot{\varphi}_j(0) = 0.$$

Therefore

$$\varphi(t) = \sum_{j=1}^{\infty}\varphi_j(t) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (28)$$

is the solution of (14) and satisfies the conditions (15), and

$$\theta(t) = \varphi_0(t) + \varphi(t) = \sum_{j=0}^{\infty}\varphi_j(t) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (29)$$

is the required solution.

Here the solution $\theta(t)$ of (9) reduces to $\varphi_0(t)$ when $\ddot{y}(t)\equiv 0$; that is to say, $\varphi_0(t)$ shows the influence of the acceleration acting on the supporting axis of the pendulum when only the x-component of it is taken into account, while $\varphi(t)$ exhibits the additional influence due to the introduction of the y-component of the acceleration.

§ 4. When t is large

If $a > 0$, the real parts of α_1, α_2 are always negative, and $|e^{\alpha_1 t}|$, $|e^{\alpha_2 t}|$ and $|e^{-at}|$ are very small when t is large. Therefore we get following formula approximately

when t is very large

$$\varphi_0(t) = -K\int_0^t N(s,t)\ddot{x}(s)ds \quad \cdots\cdots \quad (30),$$

and this formula is independent of the initial conditions. We can rewrite this formula as follows:

when $a \neq b$

$$\varphi_0(t) = -Kx(t) + \frac{K}{\alpha_1 - \alpha_2}\{\alpha_2^2 e^{\alpha_2 t}\int_0^t e^{-\alpha_2 s}x(s)ds - \alpha_1^2 e^{\alpha_1 t}\int_0^t e^{-\alpha_1 s}x(s)ds\} \quad \cdots \quad (30)',$$

when $a = b$

$$\varphi_0(t) = -Kx(t) + Kae^{-at}\{(2-at)\int_0^t e^{as}x(s)ds + a\int_0^t e^{as}sx(s)ds\} \quad \cdots\cdots \quad (30)''.$$

If the forms of $x(t)$ and $y(t)$ or $\ddot{x}(t)$ and $\ddot{y}(t)$ are given, we can perform the above integrations and can put the terms which contain $e^{\alpha_1 t}$, $e^{\alpha_2 t}$ or e^{-at} to zero in the resulting formulas.

The same procedure can be done in the resulting formulas from the calculation of

$$\varphi_j(t) - K\int_0^t N(s,t)\ddot{y}(s)\varphi_{j-1}(s)ds \quad \cdots\cdots \quad (21).$$

As the majorant series of the right side of (27) are the expansion of some exponential functions, from this inequality we can not know the behavior of $\sum\varphi_j(t)$ at $t \to \infty$. Accordingly we consider this point in the following.

Put

$$I = \int_0^t |N(s,t)|ds, \quad 0 \leq s \leq t \leq \infty,$$

then we have:

(i) **when** $a > b$

putting $\sqrt{a^2 - b^2} = m > 0$ gives

$$N(s,t) = \frac{1}{m}e^{-a(t-s)}\sinh m(t-s) \geq 0$$

therefore

$$I = \frac{1}{m}\int_0^t e^{-a(t-s)}\sinh m(t-s)ds = \frac{1}{m}\int_0^t e^{-au}\sinh mu\,du \leq \frac{1}{m}\int_0^\infty e^{-au}\sinh mu\,du = \frac{1}{b}.$$

(ii) **when** $a = b$

in this case,

$$N(s,t) = (t-s)e^{-a(t-s)} \geq 0$$

therefore

$$I=\int_0^t (t-s)e^{-a(t-s)}ds=\int_0^t ue^{-au}du \leqq \int_0^\infty ue^{-au}du=\frac{1}{a^2}=\frac{1}{b^2}$$

(iii) when $a<b$

putting $\sqrt{b^2-a^2}=m'>0$ gives

$$N(s,t)=\frac{1}{m'}e^{-a(t-s)}\sin m'(t-s)$$

and the sign of this formula may be positive or negative and

$$\begin{aligned} I&=\frac{1}{m'}\int_0^t e^{-a(t-s)}|\sin m'(t-s)|ds \\ &=\frac{1}{m'}\int_0^t e^{-au}|\sin m'u|du \leqq \frac{1}{m'}\int_0^\infty e^{-au}|\sin m'u|du. \end{aligned}$$

Now, put

$$u_j=j\pi/m' \text{ and } I_j=(-1)^j\frac{1}{m'}\int_{u_j}^{u_{j+1}} e^{-au}\sin m'u\,du$$

$$j=0, 1, 2, 3, \cdots\cdots\cdots\cdots$$

then we have

$$\frac{1}{m'}\int_0^\infty e^{-au}|\sin m'u|du=I_0+I_1+I_2+\cdots\cdots\cdots\cdots\cdots\cdots.$$

While, putting $u=u_j+v$ gives

$$\begin{aligned} I_j&=(-1)^j\frac{1}{m'}\int_0^{u_1} e^{-a(u_j+v)}\sin(j\pi+m'v)dv \\ &=\frac{1}{m'}e^{-au_j}\int_0^{u_1} e^{-av}\sin m'v\,dv=\frac{1}{b^2}(1+e^{-au_1})e^{-au_j} \end{aligned}$$

and

$$I \leqq \frac{1}{b^2}(1+e^{-au_1})\sum_{j=0}^\infty e^{-au_j}=\frac{1}{b^2}\cdot\frac{1+e^{-au_1}}{1-e^{-au_1}}.$$

Accordingly we have

$$I=\int_0^t |N(s,t)|ds \leqq k^2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (31)$$

$$0 < t \leqq \infty$$

where

$$\left.\begin{aligned} k^2&=\frac{1}{b^2} \cdots\cdots\cdots\cdots\cdots\cdots \text{ for } a \geqq b \\ k^2&=\frac{1}{b^2}\cdot\frac{1+e^{-au_1}}{1-e^{-au_1}} \cdots\cdots \text{ for } a<b \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (32)$$

Next, in (18) or (18'),

$$\left|\frac{K'\dot{x}(0)}{\alpha_1-\alpha_2}(e^{\alpha_2 t}-e^{\alpha_1 t})\right| \text{ or } |-K'\dot{x}(0)te^{-at}|$$

are both finite when $0\leqq t\leqq\infty$, and also

$$\left|\int_0^t N(s,t)\ddot{x}(s)ds\right| < G_x\int_0^t |N(s,t)|ds\leqq G_x k^2,$$

hence there exists a constant G_0 such that

$$|\varphi_0(t)| < G_0,\quad 0\leqq t\leqq\infty.$$

Then, from the equation (21), we have

$$|\varphi_1(t)|\leqq K\int_0^t |N(s,t)\ddot{y}(s)\varphi_0(s)|ds < KG_yG_0\int_0^t |N(s,t)|ds\leqq G_0\bar{G}$$

where $\bar{G}=KG_yk^2$

$$|\varphi_2(t)| < KG_yG_0\bar{G}\int_0^t |N(s,t)|ds < G_0\bar{G}^2$$

....

....

generally

$$|\varphi_j(t)| < G_0\bar{G}^j \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (33).$$

Therefore, if

$$\bar{G}=KG_yk^2 < 1 \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (34)$$

we get

$$\sum_{j=1}^{\infty}|\varphi_j(t)| < \sum_{j=1}^{\infty}G_0\bar{G}^j=\frac{G_0\bar{G}}{1-\bar{G}} \quad\cdots\cdots\cdots\cdots\cdots\cdots \quad (35)$$

independently of t, hence $\varphi(t)=\sum_{j=1}^{\infty}\varphi_j(t)$ is bounded when $0\leqq t\leqq\infty$.

In the preceding section, the uniform convergency of $\sum\varphi_j(t)$ is proved under the condition that t is finite, but the behavior of this series at $t\to\infty$ was not known. In the above discussions, the boundness at $t\to\infty$ is proved under the conditions $a>0$ and (34). Here we should take notice that the condition (34) is a sufficient condition for the boundness of the series provided that $a>0$, therefore it may not be a necessary condition. But the author believes that (34) will be not so far away from the necessary condition, for example, consider a case that always $b^2<K\ddot{y}(t)$ which does not satisfy the condition (34), then the differential equation (9) becomes non oscillatory and its

solution can increase indefinitely as t tends to ∞, and this can not be allowed under the condition that $|\theta|$ is small.

At the end of this section, we consider an example. Let

$$x(t)=A\sin nt,\ \ y(t)=B\sin nt \cdots\cdots \tag{i}$$

then from (30) and (21), we get

$$\varphi_0(t)=-U\cos(nt-\sigma_1) \cdots\cdots \tag{ii}$$

$$\left.\begin{aligned}
\varphi_1(t)&=\frac{UV}{2}\left\{\frac{1}{b^2}\sin\sigma_1-\frac{1}{\Delta_2}\cos(2nt-\sigma_1-\sigma_2)\right\}\\
\varphi_2(t)&=\frac{UV^2}{2}\left\{\frac{1}{b^2\Delta_1}\sin\sigma_1\cos(nt-\sigma_1)+\frac{1}{2\Delta_1\Delta_2}\cos(nt-2\sigma_1-\sigma_2)\right.\\
&\qquad\left.-\frac{1}{2\Delta_2\Delta_3}\cos(3nt-\sigma_1-\sigma_2-\sigma_3)\right\}\\
\varphi_3(t)&=\frac{UV^3}{4}\left[\frac{\sin\sigma_1}{b^2\Delta_1}\left\{\frac{-\sin\sigma_1}{b^2}+\frac{1}{\Delta_2}\cos(2nt-\sigma_1-\sigma_2)\right\}\right.\\
&\quad+\frac{1}{2\Delta_1\Delta_2}\left\{\frac{-1}{b^2}\sin(2\sigma_1+\sigma_2)+\frac{1}{\Delta_2}\cos(2nt-2\sigma_1-2\sigma_2)\right\}\\
&\quad+\frac{1}{2\Delta_2\Delta_3}\left\{\frac{1}{\Delta_2}\cos(2nt-\sigma_1-2\sigma_2-\sigma_3)\right.\\
&\qquad\left.\left.-\frac{1}{\Delta_4}\cos(4nt-\sigma_1-\sigma_2-\sigma_3-\sigma_4)\right\}\right]\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\cdots\cdots \tag{iii}$$

where

$$\Delta_j=\sqrt{(j^2n^2-b^2)^2+(2jan)^2},\ \ \sin\sigma_j=\frac{j^2n^2-b^2}{\Delta_j},\ \ \cos\sigma_j=\frac{2jan}{\Delta_j}$$

$$U=\frac{KAn^2}{\Delta_1},\ \ V=KBn,\ \ j=1,2,3,\cdots\cdots.$$

For example, if

$$B=0.01^{cm},\quad n=2\pi\times 15/sec,\quad b=2\pi\times 3/sec$$

$$h=a/b=0.2,\ \ K=1/2.76^{cm},$$

we have

$$KB=0.00363,\qquad KBn^2/b^2=0.0907$$

$$\Delta_j\doteqdot(25j^2-1)b^2,\quad \sigma_j\doteqdot\frac{\pi}{2},\quad U\doteqdot\frac{25}{24}KA$$

and

$$\varphi_0(t) = -U sin\ nt$$

$$\varphi_1(t) = 0.0453U + 0.00046U cos\ 2nt$$

$$\varphi_2(t) = 0.00017U sin\ nt$$

$$\varphi_3(t) = -0.000008U$$

$$\cdots\cdots\cdots\cdots\cdots\cdots .$$

therefore

$$\varphi(t) \doteqdot 0.0453U + 0.00046U cos 2nt + 0.00017U sin\ nt \doteqdot 0.0453U = const.$$

For the measurement of displacement, it is taken that $a \leqq b << n$ as is well known, and if we neglect $O\left(\frac{b^2}{n^2}\right)$ as compared with unity, we obtain

$$\varDelta_j = j^2 n^2, \quad \sigma_j = \frac{\pi}{2}, \quad U = KA$$

and

$$\varphi_0(t) = -KA sin\ nt, \quad \varphi_1(t) = KA \cdot \frac{KBn^2}{2b^2}$$

$$\varphi_2(t) = P\varphi_0(t), \qquad \varphi_3(t) = P\varphi_1(t)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

generally

$$\varphi_{2r}(t) = P^r \varphi_0(t), \quad \varphi_{2r+1}(t) = P^r \varphi_1(t) \quad \cdots\cdots\cdots\cdots \quad \text{(iv)}$$

$$r = 0, 1, 2, \cdots\cdots\cdots\cdots$$

where

$$P = -K^2B^2n^2/2b^2 \quad \cdots\cdots\cdots\cdots \quad \text{(v)}.$$

Therefore if $|P| < 1$, that is

$$KB < \frac{\sqrt{2}\,b}{n} \quad \cdots\cdots\cdots\cdots \quad \text{(vi)},$$

we get

$$\varphi(t) = \sum_{j=1}^{\infty} \varphi_j(t) = \varphi_0(t) \sum_{r=1}^{\infty} P^r + \varphi_1(t) \sum_{r=0}^{\infty} P^r$$

$$= \frac{1}{1-P} \{P\varphi_0(t) + \varphi_1(t)\}$$

that is

$$\left.\begin{aligned}\varphi(t)&=KA\cdot\frac{KB}{\frac{2b^2}{n^2}+K^2B^2}(1+KB\sin nt)\\&\fallingdotseq KA\cdot\frac{KB}{\frac{2b^2}{n^2}+K^2B^2}=const\end{aligned}\right\}\qquad\cdots\cdots\cdots\cdots\text{(vii)}.$$

We assumed that b/n is very small and $O(b^2/n^2)$ is negligible as compared with unity, then from (vi), BK is very small also. But only by these conditions, we can not assert that $\beta=BK/\left(\frac{2b^2}{n^2}+B^2K^2\right)<1$. For example, if $n=2\pi\times10/sec$ and if an inverted pendulum with its constants $b=2\pi\times\frac{1}{6}/sec$, $K=1/30^{cm}$ is used, we get

when $B=0.03^{cm}$: $BK=10^{-3}$, $b/n=1/60$ $\therefore$ $\beta\fallingdotseq1.8$

when $B=0.1^{cm}$: $BK=1/300$, $b/n=1/60$ $\therefore$ $\beta\fallingdotseq6$.

Generally, put $b/n=\varepsilon$ and $BK=O(\varepsilon^p)$, then it is evident that $p\geqq1$ form (vi), but if $1\leqq p<2$ we have $\beta=O(\varepsilon^{p-2})$ and $p-2<0$, therefore β is very large.

The above discussion is stated under the condition $a>0$. If $a=0$ (of course a is not negative), that is, if the damping term vanishes, we get

$$\varphi_0(t)=-Kx(t)+Kx(0)\cos bt+\frac{K-K'}{b}\dot{x}(0)\sin bt$$
$$+Kb\int_0^t\sin b(t-s)x(s)ds\quad\cdots\cdots\quad(18)''$$

from (18) or (19) and

$$N(s,t)=\frac{1}{b}\sin b(t-s)\quad\cdots\cdots\cdots\cdots\quad(17)''$$

and we can not know the boundness of

$$\int_0^t\sin b(t-s)x(s)ds,\quad O\leqq t\leqq\infty$$

even if $|x(t)|$ is bounded, and moreover

$$\int^t|N(s,t)|ds=\infty$$

is obtained. Dr. K. SUEHIRO discussed a special case of this case $(a=0)$.*

§ 5. Initial state

When we consider the initial state, $|t|$ is very small and we can express $\theta(t)$

* See references.

by a power series of t. Putting

$$\theta(t)=\theta(0)+t\dot{\theta}(0)+\frac{t^2}{2}\ddot{\theta}(0)+\frac{t^3}{3!}\dddot{\theta}(0)+\cdots\cdots\cdots\cdots$$

and calculating $\theta(0)$, $\dot{\theta}(0)$, $\ddot{\theta}(0)$, $\cdots\cdots$, we get

$$\begin{aligned}\theta(t) &= t\{-K'\dot{x}(0)\}+\frac{t^2}{2}\{2aK'\dot{x}(0)-K\ddot{x}(0)\}\\ &+\frac{t^3}{3!}[\{b^2-4a^2-K\ddot{y}(0)\}K'\dot{x}(0)+2aK\ddot{x}(0)-K\dddot{x}(0)]\\ &+\frac{t^4}{4!}[\{8a^3-4ab^2+4aK\ddot{y}(0)-2K\dddot{y}(0)\}K'\dot{x}(0)\\ &\qquad+\{b^2-4a^2-K\ddot{y}(0)\}K\ddot{x}(0)+2aK'\dddot{x}(0)-K'\ddddot{x}(0)]\\ &+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (36).\end{aligned}$$

In this formula, $\ddot{y}(0)$ appears for the first time in the coefficient of t^3 when $\dot{x}(0)\neq 0$; $\ddot{y}(0)$ appears for the first time in the coefficient of t^4 when $\dot{x}(0)=0$, $\ddot{x}(0)\neq 0$ and so on. And generally, we can show that if the first term of the expansion of $\theta(t)$ is $O(t^p)$, $\ddot{y}(0)$ appears for the first time in the coefficient of t^{p+2}, hence the initial state of $\theta(t)$ does not almost be affected by the existence of $y(t)$.

§ 6. Comments

The series expressed by (29) converges uniformly and is a continuous function of t in the interval $O\leq t<\infty$, and also is the accurate solution of the equation (9) satisfying the conditions (10), and it is proved that it is always bounded when $O\leq t\leq\infty$ under the conditions $G<1$, and $a>0$. When $\bar{G}=KG_yk^2$ is very small, the series (29) converges very rapidly, and practically in order to get the good approximation of $\theta(t)$, it is sufficient to take first few terms of this series as shown in the example at the end of § 4. But when $\bar{G}=KG_yk^2$ is not so small, the convergency of the series is so slow that it seems worthless for the practical use, and the direct calculation of the numerical solution of (9) by a well known method seems to be more practical than the term by term-calculation of the series.

References

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ON THE BALLOONING PROBLEM (IV)
(Capedge Friction)

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10. Capedge Friction

Next we will study the problems concerning with the connection of the outer and inner balloons at the capedge. There are two kinds of frictional force acting at this point, namely, a force acting along the rim curve of the edge which makes the value of ω small, and other along the thread which hinder the sending velocity of the yarn.

In Fig. 7 RR′ is the rim curve, and each of A′A and AA″ is a part of outer and inner balloon respectively.

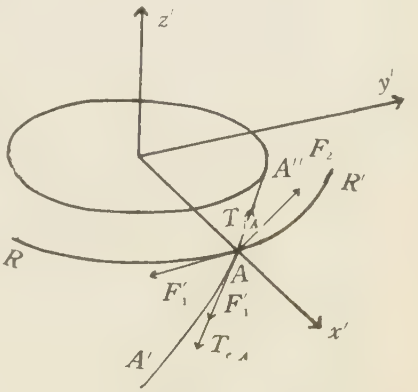

Fig. 7.

All forces which are in equilibrium at A are:

$T_{i,A}$ tension of inner balloon at A,

$T_{e,A}$ tension of outer balloon at A,

F_1' frictional force along the rim curve, evidently parallel to the negative y' axis,

F_1'' frictional force parallel to $T_{e,A}$,

F_2 reactional force of the edge surface upon the thread at A. All the direction cosines of these forces with respect to x', y', z' axis (Fig. 5) are immediately given as follows. Direction of $T_{e,A}$ and F_1'' are

$$\left(-\frac{dx'}{ds},\ -\frac{dy'}{ds},\ -\frac{dz'}{ds}\right)=\left(-\cos\theta_A\left(\frac{dx}{ds}\right)_A-\sin\theta\left(\frac{dy}{ds}\right)_A,\right.$$

$$\left.\sin\theta_A\left(\frac{dx}{ds}\right)_A-\cos\theta_A\left(\frac{dy}{ds}\right)_A,\ -\left(\frac{dz}{ds}\right)_A\right)=\left(-\frac{c}{a}\frac{T_0}{T_{e,A}}\cos\alpha_0\left(\frac{d\rho}{dz}\right)_1,\right.$$

$$\left.-\frac{c}{a}\frac{T_0}{T_{e,A}}\cos\alpha_0\rho_1\left(\frac{d\theta}{dz}\right)_1,\ -\frac{T_0}{T_{e,A}}\cos\alpha_0\right)$$

where suffix 1 denotes the values just at A in the external ballooning solutions (so for instance ρ_1 can be put to unity). Direction of $T_{i,A}$ is simply $(-\sin\tau_{i,A},\ \cos\tau_{i,A},\ 0)$ from the definition of τ. As for the direction of F_2, we assume that F_2 lies in the plane parallel to $x'-z'$ plane, and put its direction cosines $(l',\ 0,\ n')$. Then the equations of equilibrium become

$$\left.\begin{aligned}
&-\frac{c\,cos\alpha_0 T_0}{a}\frac{(T_{e\cdot A}+F_1'')}{T_{e\cdot A}}\left(\frac{d\rho}{dz}\right)_1 - T_{i\cdot A}sin\tau_{i\cdot A}+F_2 l'=0\\
&-\frac{c\,cos\alpha_0 T_0}{a}\frac{(T_{e\cdot A}+F_1'')}{T_{e\cdot A}}\left(\frac{d\theta}{dz}\right)_1 + T_{i\cdot A}cos\tau_{i\cdot A}-F_1'=0\\
&-cos\alpha_0 T_0\frac{(T_{e\cdot A}+F_1'')}{T_{e\cdot A}}+F_2 n'=0
\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots \quad (10.1)$$

Naturally we can assume both of F_1' and F_1'' are proportional to F_2, hence we put

$$F_1' = \kappa' F_2 \quad F_1'' = \kappa'' F_2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (10.2)$$

where κ' and κ'' are two coefficients of friction.

Now from the fundamental equation of outer ballooning given in [I], it follows that

$$\frac{c\,cos\alpha_0}{a}\left(\frac{d\rho}{dz}\right)_1 = -\left(\frac{T_{e\cdot A}^2}{T_0^2}sin^2\tau_{e\cdot A}-cos_2\alpha_0\right)^{1/2}$$

$$\frac{c\,cos\alpha_0}{a}\left(\frac{d\theta}{dz}\right)_1 = \frac{T_{e\cdot A}}{T_0}cos\tau_{e\cdot A}$$

Hence (10.1) become

$$\left.\begin{aligned}
&F_2\left\{l'+\kappa''\left(sin^2\tau_{e\cdot A}-\frac{T_0^2}{T_{e\cdot A}^2}cos^2\alpha_0\right)^{1/2}\right\}\\
&\qquad\qquad = -(T_{e\cdot A}^2 sin^2\tau_{e\cdot A}-T_0^2cos^2\alpha_0)^{1/2}+T_{i\cdot A}sin\tau_{i\cdot A}\\
&F_2\{\kappa'+\kappa''cos\tau_{e\cdot A}\} = -T_{e\cdot A}cos\tau_{e\cdot A}+T_{i\cdot A}cos\tau_{i\cdot A}\\
&F_2\left\{n'-cos\alpha_0\kappa''\frac{T_0}{T_{e\cdot A}}\right\} = T_0 cos\alpha_0
\end{aligned}\right\}\cdots\cdots\cdots\cdots \quad (10.3)$$

These are the fundamental equations which determine the relation between the outer and inner ballooning.

If we eliminate F_2, l' and n' from these equations, we get a formula which expresses $\frac{T_{i\cdot A}}{T_0}$ in terms of $T_{e\cdot A}$, $\tau_{e\cdot A}$ and other known quantities. Process of computation is rather tedious, but in the following one way of this computation is shown, which is considered to be most convenient than others.

First put

$$L = \left(sin^2\tau_{e\cdot A}-\frac{cos^2\alpha_0}{T_{1A}^2}\right)^{1/2}$$

$$M = x\,cos\tau_{i\cdot A}-T_{1\cdot A}cos\tau_{e\cdot A}$$

$$N = \kappa'+\kappa''cos\tau_{e\cdot A}$$

$$x = T_{i\cdot A}/T_0$$

$$T_{1\cdot A} = T_{e\cdot A}/T_0$$

Then (10.3) will become

$$\frac{F_2}{T_0} = \frac{x\sin\tau_{i\cdot A} - T_{1A}L}{l' + \kappa'' L} = \frac{M}{N} = \frac{\cos\alpha_0}{n' - \dfrac{\kappa''\cos\alpha_0}{T_{1A}}} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (10.4)$$

Using $l'^2 + n'^2 = 1$, it is easily obtained that

$$M^2 = (Nx\sin\tau_{i\cdot A} - LQ)^2 + \frac{\cos^2\alpha_0}{T_{1A}^2} Q^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (10.5)$$

where

$$Q = M\kappa' + NT_{1A} = x\kappa\cos\tau_{i\cdot A} - T_{1A}\kappa$$

(10.5) is essentially an algebraic quadratic equation for x, which is rewritten in the form

$$Ax^2 - 2Bx + C = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (10.6)$$

where

$$A = A' - A'' + A'''$$
$$A' = \cos\tau_{i\cdot A}(1 - \kappa' \sin\tau_{i\cdot A})$$
$$A'' = (N\sin\tau_{i\cdot A})^2$$
$$A''' = 2NL\kappa''\sin\tau_{i\cdot A}\cos\tau_{i\cdot A}$$
$$B = T_{1A}B_0$$
$$B_0 = B' - B''$$
$$B' = \cos\tau_{i\cdot A}(\cos\tau_{e\cdot A} + \kappa'\kappa''\sin^2\tau_{e\cdot A})$$
$$B'' = NL\kappa'\sin\tau_{i\cdot A}$$
$$C = T_{1A}C_0$$
$$C_0 = \cos\tau_{e\cdot A} - \kappa \sin\tau_{e\cdot A}$$

Value of $\cos\tau_{i\cdot A}$ contained in the right side of above formulas can be replaced by b/c on account of the results obtained in [III], showing that shape of the inner balloon is very nearly straight line. For T_{1A} and $\tau_{e\cdot A}$ on the other hand, i. e. the tension and the angle τ of outer ballooning just at A, we use the value which were

Table 12.

	m_{75} *emp*	m_{75} *full*	m_{150} *emp*	m_{150} *full*
$\omega_b(r.p.m)$	$6.00\cdot10^3$	$6.00\cdot10^3$	$5.75\cdot10^3$	$5.75\cdot10^3$
$T_0(gr)$	6.0	2.7	6.5	9.0
$\cos\alpha_0$	0.9636	0.9063	0.9205	0.8572
T_{1A}	0.9912	1.2727	1.0127	1.1519
$\cos\tau_{e\cdot A}$	0.2296	0.6529	0.3969	0.6489
$\cos\tau_{i\cdot A}$	0.4135	0.7372	0.4135	0.7372
L	0.04673	0.2582	0.1277	0.1584

numerically obtained in [II]. Thus we can compute each coefficient A, B and C for given κ', κ'' on which we assume several reasonable values. Results of computation are tabulated in Table 12 and 13.

Table 13. (A) m_{75}

κ'	κ''	$T_{i\cdot A}/T_0$ emp	$T_{i\cdot A}/T_0$ full	l' emp	l' full	n' emp	n' full	F_2/T_0 emp	F_2/T_0 full
0.0	0.0	0.5505	1.1272	0.427	0.426	0.904	0.905	1.065	1.004
0.0	0.1	0.6169	1.2238	0.427	0.431	0.904	0.902	1.194	1.091
0.0	0.2	0.7014	1.3383	0.427	0.431	0.904	0.903	1.358	1.192
0.1	0.0	0.8414	1.2696	0.598	0.504	0.801	0.863	1.203	1.050
0.1	0.1	0.9577	1.3837	0.598	0.504	0.801	0.863	1.369	1.144
0.1	0.2	1.1111	1.5203	0.598	0.504	0.801	0.863	1.588	1.257
0.2	0.0	1.2574	1.4274	0.752	0.574	0.659	0.819	1.461	1.107
0.2	0.1	1.4749	1.5636	0.752	0.574	0.659	0.819	1.714	1.213
0.2	0.2	1.7832	1.7283	0.752	0.574	0.659	0.819	2.072	1.341

Table 13. (B) m_{150}

κ'	κ''	$T_{i\cdot A}/T_0$ emp	$T_{i\cdot A}/T_0$ full	l' emp	l' full	n' emp	n' full	F_2/T_0 emp	F_2/T_0 full
0.0	0.0	0.9719	1.0140	0.635	0.505	0.773	0.862	1.191	0.994
0.0	0.1	1.1019	1.1098	0.635	0.505	0.773	0.861	1.350	1.089
0.0	0.2	1.2714	1.2255	0.635	0.505	0.773	0.862	1.558	1.201
0.1	0.0	1.3130	1.1559	0.757	0.572	0.653	0.819	1.409	1.047
0.1	0.1	1.5252	1.2711	0.757	0.572	0.653	0.820	1.636	1.149
0.1	0.2	1.8191	1.4121	0.757	0.572	0.653	0.820	1.952	1.277
0.2	0.0	1.8442	1.3153	0.860	0.636	0.511	0.772	1.803	1.111
0.2	0.1	2.2439	1.4556	0.860	0.636	0.511	0.772	2.194	1.229
0.2	0.2	2.8638	1.6295	0.860	0.636	0.511	0.772	2.799	1.376

§ 11. Connection of outer and inner balloon.

Now we have come to the last and most important problem of connecting the solutions obtained already in the former chapters. In [I] and [II] we have discussed solely the outer ballooning and have shown that the tension of every part of the yarn and the shape of balloon will be wholly determined provided the tension of the top guide T_0 is given. In [III] the same was discussed about the inner ballooning, while in [IV] the questions regarding to the connection of both balloons at the cap edge have been studied, and a formula was proposed which gives $T_{i\cdot A}/T_0$ for several assumed values of edge friction coefficients. Summarising all these results of each solutions synthetically, we are now able to compute T_D, $T_{i\cdot A}$, $T_{e\cdot A}$ if a single value of T_0 is given. In other words the ratio's T_D/T_0, $T_{i\cdot A}/T_0$, $T_{e\cdot A}/T_0$ wholly determine the shape of both balloons together with the tension at every points of the yarn.

What desides an absolute values of tension of the yarn must be some unknown friction such as a force acting between the wund yarn and the bobbin for instance.

To know such relation the caluclation of the complicating energy equation is required, which however, is no practicable use in the actual problem. The most practicable way to determine such absolute values is no doubt to measure T_0 directly and insert it in the equations obtained above. Values of κ' and κ'' are difficult to measure in the actual works, and several preliminary experiments which have been done up to present have shown that these values lie between 0.1 and 0.2, and it is thought that 0.15 is not so much different from its real value. Of course these depend upon the state of edge surface greatly and often become a cause of sudden change of yarn tension.

In table 14, relative values of tension are tabulated against T_0 for a case of 75 denier, from which one can see easily the range of variation of the tension according to the surface edge friction.

Table 14. m_{75}

κ'	κ''	$T_{e \cdot A}/T_0$		T_{iA}/T_0		T_D/T_0	
		emp	*full*	*emp*	*full*	*emp*	*full*
0.0	0.0	0.991	1.273	0.550	1.127	0.564	1.168
0.0	0.1	0.991	1.273	0.617	1.224	0.632	1.268
0.0	0.2	0.991	1.273	0.701	1.338	0.719	1.386
0.1	0.0	0.991	1.273	0.841	1.270	0.862	1.316
0.1	0.1	0.991	1.273	0.958	1.384	0.982	1.434
0.1	0.2	0.991	1.273	1.111	1.520	1.139	1.575
0.2	0.0	0.991	1.273	1.257	1.427	1.288	1.478
0.2	0.1	0.991	1.273	1.475	1.565	1.512	1.620
0.2	0.2	0.991	1.273	1.783	1.728	1.828	1.790

ON THE STATIONARY FLOW OF VAPOR

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1. Introduction

The rate of vaporisation from the surface of a liquid depends on (1) the temperatures of the liquid and the gas near the surface of the liquid, (2) the pressure acted on the surface of the liquid, (3) the state of the flow of gas contiguous to the surface of the liquid and (4) the vapor pressure near the surface of the liquid. But it is very difficult to acquire theoretically some exact relations between the rate of vaporisation and the quantities above mentioned.

We will take a special ideal case in the following and study theoretically the rate of vaporisation in that case. By the results we may understand the important properties of the rate of vaporisation and we may apply the results effectively to the practical case of the vaporisation.

2. A Method to Obtain the Stationary Flow of Vapor

Fig. 1 shows the plan for obtaining the stationary flow of the vapor. A and B are the heat reservoirs whose temperatures are fixed at $T_1^0 K$ and $T_2^0 K$ respectively. $(T_1 \quad T)$.

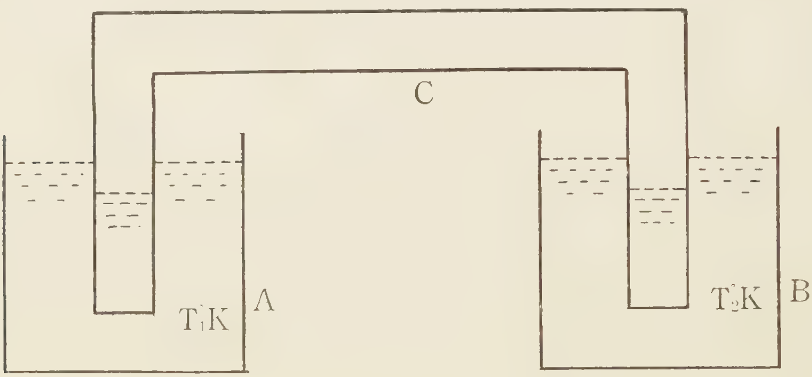

Fig. 1.

C is a revers U glass or metal tube whose middle part is placed horizontally and whose legs are fixed vertically.

The both ends of the tube are closed and the two legs are put in the heat reservoirs. A certain liquid is put in the tube, a part of the liquid in the leg in the

heat reservoir A, the other part of the liquid in the leg in the heat reservoir B. The exposed part of the tube C is covered with some heat insulator. The pressure in the tube is regulated to a proper value. The vapor evaporates in the leg of the temperature T_1^0K, flows through the tube and condenses in the leg of the temperature T_2^0K. After a while the flow of vapor evaporates nearly stationary. In the leg of the temperature T_1^0K, the vapor evaporates and absorbs heat, and in the leg of the temperature T_2^0K, it gives out heat and condenses.

In this case, the rate of vaporisation depends on the values of T_1, T_2, on the area of the surfaces of the liquid in the two legs, on the sectional area and the length of the tube and on the pressure of the foreign gas contained in the tube. The relations of the rate of vaporisation with these quantities above mentioned, will be obtained experimentally by measuring the rate of vaporisation under various conditions. We constructed practically such a simple apparatus as Fig. 1 and measured the rates of vaporisation of some volatile liquids under some conditions and obtained preliminary informations about it. In the forthcoming paper we will study the rate of vaporisation under various exact conditions.

3. The Thermodynamical Relations referring to the Stationary Flow of the Vapor

The slow stationary flow of the vapor in a tube, whose wall is constructed by some heat insulator, can be looked upon as the flow of constant enthalpy, because the flow of heat through the wall of the tube is prohibited.

Now we express the internal energy, the enthalpy and the volume of the unit mass of the liquid at the temperature T_1^0K by U_1, H_1 and V_1 respectively, and those of the saturated vapor at T_1^0K by U_1', H_1' and V_1' respectively, and the pressure of the saturated vapor by p_1. We express those of the liquid at T_2^0K by U_2, H_2 and V_2 respectively and those of the saturated vapor by U_2', H_2' and V_2' respectively and the pressure of the saturated vapor by p_2, as shown in Fig. 2.

Along the flow of the vapor the thermodynamical quantities of the unit mass of the vapor will change generally. We express the internal energy, the enthalpy, and

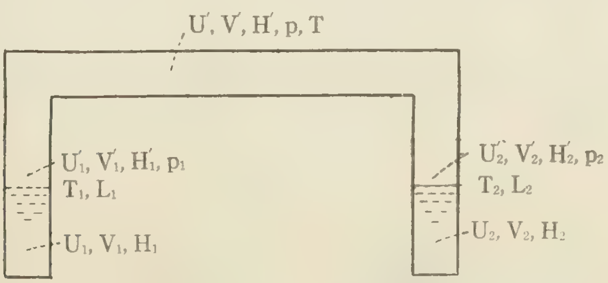

Fig. 2.

the volume of the unit mass of vapor at any section of the flow by U', H' and V' respectively, and the temperature and the pressure at that section by T^0K and p respectively. These quantities will depend on the position of the section. We express the latent heats of vaporisation at T_1^0K and T_2^0K by L_1 and L_2 respectively.
Then the relations

$$L_1=U_1'-U_1+P_1(V_1'-V_1)=H_1'-H_1,$$
$$L_2=U_2'-U_2+P_2(V_2'-V_2)=H_2'-H_2 \quad\cdots\cdots\cdots\cdots\cdots\cdots\quad (3.1)$$

hold. The pressure of the vapor will decrease and the volume of the unit mass of the vapor will increase along the flow, but the enthalpy will not change, and therefore we can put $H'=H_1'$. The temperature of the vapor will change slightly, but if we can take the vapor approximately as a perfect gas, we can take the temperature as constant along the flow, i. e. $T=T_1$, and the vapor will gradually become unsaturated. If we can not apply the law of perfect gas to the vapor, T will be slightly different from T_1, in many cases $T<T_1$. But at the condensed end p will be equal to p_2 and the heat moves from vapor to liquid and the vapor will become saturated at the temperature T_2^0K. The enthalpy of the vapor becomes H_2'. Accordingly if the specific heat at constant pressure of the vapor is C_p',

$$H'-H_2'=H_1'-H_2'=\int_{T_2}^{T} C_p' dT \quad\cdots\cdots\cdots\cdots\cdots\cdots\quad (3.2)$$

If the vapor can be taken as a perfect gas and C_p' is constant in the range of the tepmerature T_1 to T_2, (3.2) becomes

$$H'-H_2'=H_1'-H_2'=C_p'(T-T_2)=C_p'(T_1-T_2)$$

In the liquid, which is in the state of equilibrium with its saturated vapor, the change of the enthalpy dH, caused by the change of the temperature dT and by the change of the pressure dp is expressed by

$$dH=C_p dT+\left\{V-T\left(\frac{\partial V}{\partial T}\right)_p\right\}dp$$
$$=C_p dT+V(1-T\beta)\,dp \quad\cdots\cdots\cdots\cdots\cdots\cdots\quad (3.3)$$

where V is the volume of the unit mass of the liquid, β is the coefficient of expansion of the liquid and C_p is the specific heat at constant pressure of the liquid.
There is the relation

$$\frac{dp}{dT}=\frac{L}{T(V'-V)} \quad\cdots\cdots\cdots\cdots\cdots\cdots\quad (3.4)$$

where V' is the volume of the unit mass of the saturated vapor and L is the heat of vaporisation at the temperature T^0K.
Accordingly (3.3) becomes

$$\frac{dH}{dT}=C_p+\left\{V-T\left(\frac{\partial V}{\partial T}\right)_p\right\}\frac{L}{T(V'-V)}$$

$$\doteq C_p+\frac{LV}{V'}\left(\frac{1}{T}-\beta\right) \qquad (3.5)$$

By (3.3) and (3.5) we get

$$H_1-H_2=\int_{T_2}^{T_1}C_p dT+\int_{p_2}^{p_1}V(1-T\beta)dp$$

and

$$H_1-H_2=\int_{T_2}^{T_1}\left\{C_p+\frac{LV}{V'}\left(\frac{1}{T}-\beta\right)\right\}dT. \qquad (3.6)$$

We get, therefore, the relation

$$L_2-L_1=H_1-H_2-(H'_1-H'_2)$$

$$=\int_{T_2}^{T_1}\left\{(C_p-C'_p)+\frac{LV}{V'}\left(\frac{1}{T}-\beta\right)\right\}dT. \qquad (3.7)$$

Also (3.7) will be obtained by the following considerations.
The unit mass of the liquid absorbs the heat L_1 at the T_1 end and becomes the saturated vapor. The vapor flows through the tube as the unsaturated vapor to the T_2 end and gives out the heat $\int_{T_2}^{T_1}C'_p dT$ and becomes saturated. The vapor gives out the heat of vaporisation L_2 and becomes liquid at T_2. Accordingly we get the relation

$$H_1-H_2=L_2+\int_{T_2}^{T_1}C'_p dT-L_1$$

which is equal to (3.7).

Thus, through the transfer of the vapor of unit mass, the heat L_1 is absorbed from the heat reservoir at $T_1^{\circ}K$ and the heat $L_2+\int_{T_2}^{T_1}C'_p dT$ is given to the heat reservoir at $T_2^{\circ}K$.

As the value of $\frac{LV}{V'}\left(\frac{1}{T}-\beta\right)$ is very small compared with C_p in many cases, we may use

$$H_1-H_2=\int_{T_2}^{T_1}C_p dT$$

instead of (3.6). Accordingly, (3.7) may be expressed in the simpler form,

$$L_2-L_1=\int_{T_2}^{T_1}(C_p-C'_p)dT \qquad (3.8)$$

which is the well-known Kirchhoff's equation*).

The increase of the entropy, caused by this stationary flow of the vapor, is

*) This equation is usually used in the case of the sublimation.

considered as the sum of the increase of the entropy of the reservoirs $\triangle S_1$, and the increase of the entropy of the liquid $\triangle S_2$. $\triangle S_1$ is expressed by

$$\triangle S_1 = \frac{1}{T_2}\left(L_2 + \int_{T_2}^{T_1} C'_p dT\right) - \frac{L_1}{T_1}. \quad \cdots\cdots (3.\ 9)$$

To obtain $\triangle S_2$, we use the relation

$$dS = \frac{dH}{T} - \frac{V}{T}dp = C_p\frac{dT}{T} + \frac{V(1-T\beta)}{T}dp - \frac{V}{T}dp$$
$$= C_p\frac{dT}{T} - V\beta dp = C_p\frac{dT}{T} - \left(\frac{\partial V}{\partial T}\right)_p dp.$$

This relation is transformed to

$$dS = C_p\frac{dT}{T} - \frac{L}{V'-V}\left(\frac{\partial V}{\partial T}\right)_p\frac{dT}{T},$$

by using the Clapeyron's formula. By this relation we get

$$\triangle S_2 = \int_{T_1}^{T_2}\left\{C_p - \frac{L}{V'-V}\left(\frac{\partial V}{\partial T}\right)_p\right\}\frac{dT}{T} \quad \cdots\cdots (3.10)$$

The total change of the entropy $\triangle S$ is given by

$$\triangle S = \frac{1}{T_2}\left(L_2 + \int_{T_2}^{T_1} C'_p dT\right) - \frac{L_1}{T_1} - \int_{T_2}^{T_1}\left\{C_p - \frac{L}{V'-V}\left(\frac{\partial V}{\partial T}\right)_p\right\}\frac{dT}{T} \quad \cdots\cdots (3.11)$$

In many cases, as $C_p \gg \frac{L}{V'-V}\left(\frac{\partial V}{\partial T}\right)_p$, $\triangle S$ may be approximated by

$$\triangle S = \frac{1}{T_2}\left(L_2 + \int_{T_2}^{T_1} C'_p dT\right) - \frac{L_1}{T_1} - \int_{T_2}^{T_1} C_p\frac{dT}{T}. \quad \cdots\cdots (3.11')$$

4. The Rate of Flow of the Vapor

The quantity of heat transferred and the entropy change per unit time are determined, if the rate of flow of the vaper is determined. We study the rate of flow of the vapor. As the flow of the vapor occurs usually in another non-condensable gas, we consider the case where the water vapor or the vapor of other liquid flows stationary through the air or another gas of some pressure.

We take z-axis in the direction of the flow, and express the molecular densities of the vapor and the gas at the place z by n_1 and n_2, the diffusion constants of the vapor and the gas by D_1 and D_2 respectively.

If we take the case where the flow of the fluid, as the whole mixture of the vapor and the gas, do not occurs, the number of molecules of the vapor and the gas transferred by diffusion through the unit area perpendicular to the z direction per unit time are given by[1]

$$\Gamma_1 = -D_1 \frac{dn_1}{dz}, \quad D_1 = \frac{1}{3} l_1 \bar{v}_1 = \frac{4}{3\pi} \frac{\bar{v}_1}{4n_1 d_1^2 + n_2(d_1+d_2)^2} \quad \cdots\cdots\cdots\cdots \quad (4.1)$$

and

$$\Gamma_2 = -D_2 \frac{dn_2}{dz}, \quad D_2 = \frac{1}{3} l_2 \bar{v}_2 = \frac{4}{3\pi} \frac{\bar{v}_2}{4n_2 d_2^2 + n_1(d_1+d_2)^2} \quad \cdots\cdots\cdots\cdots \quad (4.2)$$

respectively, where l_1, $\bar{v}_1$ and d_1 are the mean free path, the mean velocity and the diameter of the vapor molecule, l_2, $\bar{v}_2$ and d_2 are those of the gas molecule respectively.

In the case of the stationary flow of the vapor, only the vapor does flow, and the gas maintains some fixed density gradient and the latter does not flow as a whole. In that state, it is considered that, though the gas flows with the vapor as a component of the whole mixture, its resultant flow vanishes as it is pushed back by the effect of the diffusion, while the vapor flows as another part of the flow of the mixture in z direction and flows by diffusion in the same direction.

The flow of the mixture as a whole depends on the pressure gradient or the molecular density gradient of the mixture. In some simple case we may assume that the rate of flow of the mixture is proportional to the molecular density gradient of the mixture. We consider here such a case.

If we put $n = n_1 + n_2$, the total molecular density, the number of molecules of the mixture which pass through the unit area perpendicular to z direction in the flow of the mixture as a whole, is given by

$$\Gamma = -C \frac{dn}{dz} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (4.3)$$

where C is a constant, the value of which depends on the sectional area of the tube, the viscosity and the speed of flow of the mixture.

The total number Γ_1' of vapor molecules which pass through the unit area perpendicular to z direction per unit time in the flow of the mixture and the flow due to the diffusion is given by

$$\Gamma_1' = -D_1 \frac{dn_1}{dz} + \Gamma \frac{n_1}{n} = -D_1 \frac{dn_1}{dz} - C \frac{n_1}{n} \frac{d(n_1+n_2)}{dz} \quad \cdots\cdots\cdots\cdots \quad (4.4)$$

The total number Γ_2' of the gas molecules which pass through the unit area perpendicular to the z direction in the flow of the mixture and the flow due to diffusion is 0 as follows:

$$\Gamma_2' = -D_2 \frac{dn_2}{dz} + \Gamma \frac{n_2}{n} = -D_2 \frac{dn_2}{dz} - C \frac{n_2}{n} \frac{d(n_1+n_2)}{dz} = 0 \quad \cdots\cdots \quad (4.5)$$

From (4.5) we obtain the relation between $\frac{dn_1}{dz}$ and $\frac{dn_2}{dz}$ in the stationary flow of the vapor as the following,

$$\frac{dn_2}{dz} = -\frac{Cn_2}{D_2n+Cn_2}\frac{dn_1}{dz}. \qquad (4.6)$$

By putting this relation (4.6) in (4.4), we get

$$\Gamma_1' = -\left\{D_1 + D_2\frac{Cn_1}{D_2n+Cn_2}\right\}\frac{dn_1}{dz}. \qquad (4.7)$$

The pressure gradient or the molecular density gradient of the mixture is obtained by using the relation (4.6) as follows:

$$\frac{dn}{dz} = \frac{D_2n}{D_2n+Cn_2}\frac{dn_1}{dz} \qquad (4.8)$$

The mass of the vapor which passes through the unit area perpendicular to z direction per unit time is obtained by multiplying Γ_1' with the molecular mass m, and so the quantity of heat transferred may be obtained as $mL_1\Gamma_1'$, where L_1 is the latent heat of vaporisation at T_1^0K.

To obtain the value of Γ_1', it is necessary to know the values of D_1, D_2, C and the distributions of n_1 and n_2 along z direction. If the form of the tube, the temperatures T_1^0K and T_2^0K of the heat reservoirs, the quantity and the kind of the gas and the kind of the liquid are given, the distributions of n_1 and n_2 can be determined in the stationary case. Accordingly the values of D_1, D_2 and $\frac{dn_1}{dz}$ can be determined and Γ_1' are obtained. But in practical cases, to obtain theoretically the value of Γ_1' is very difficult. Here we study only the special simple case in the following.

5. A Special Simple Case

We take a special case where the conditions, $d_1 = d_2 = d$ and $\bar{v}_1 = \bar{v}_2 = \bar{v}$ are satisfied. In this case, as we can put

$$D_1 = D_2 = \frac{\bar{v}}{3\pi nd^2} = D, \qquad (5.1)$$

we get

$$\frac{dn_2}{dz} = -\frac{Cn_2}{A+Cn_2}\frac{dn_1}{dz}. \qquad (5.2)$$

$$\frac{dn}{dz} = \frac{A}{A+Cn_2}\frac{dn_1}{dz}, \qquad (5.3)$$

where

$$A = Dn = \frac{\bar{v}}{3\pi d^2},$$

and

$$\Gamma_1' = -D\left(1+\frac{Cn_1}{A+Cn_2}\right)\frac{dn_1}{dz}$$
$$= -D\frac{(D+C)n}{A+Cn_2}\frac{dn_1}{dz} \doteqdot -D\frac{Cn}{Dn+Cn_2}\frac{dn_1}{dz}. \qquad (5.4)$$

If n_2 is not very small, (5.4) becomes

$$\Gamma_1' \doteqdot -D\frac{n}{n_2}\frac{dn_1}{dz} = -\frac{A}{n_2}\frac{dn_1}{dz} \qquad (5.4')$$

If we put $n_2 = \varepsilon n$ $(0 < \varepsilon < 1)$,

we obtain $\Gamma_1' = -\frac{A}{\varepsilon n}\frac{dn_1}{dz} = -\frac{D}{\varepsilon}\frac{dn_1}{dz}. \qquad (5.4'')$

Thus we know that Γ_1' is almost inversely proportional to the density of the gas, that is to say, it is strongly affected by the value of ε.

If n_2 is so small that $A \gg Cn_2$, then we can put $n_1 \doteqdot n$ and then taking into account of the fact that the condition $C \gg D$ is usually satisfied, we get

$$\Gamma_1' \doteqdot -C\frac{dn_1}{dz}. \qquad (5.4''')$$

If n_2 is so large that $n_2 \gg n_1$, then

$$\Gamma_1' = -D\frac{dn_1}{dz}. \qquad (5.4'''')$$

From these results, we know clearly that for the rapid vaporisation, it is very important to get low pressure as well as to get high temperature.

The value of C in the expression of Γ_1' depends upon the radius and the form of the tube, the kind of the gas and the vapor, and the velocity of the vapor flow in some oomplicated manner. It is. therefore, very difficult to obtain the value of C in the general case.

To obtain a rough estimation about the value of C, we consider one simple case where the flow of vapor can be seen as the Poiseuille's flow in a circular-cylindrical tube. If the radius of the tube, the pressure in the tube and the coefficient of viscosity are r, p and η respectively, the volume of the fluid which flows out from the tube per unit time, is expressed by

$$V = -\frac{\pi r^4}{8\eta}\frac{dp}{dz}.$$

If the fluid obeys the law of ideal gas, i. e. if $pV = NkT$, the number of the gas molecules which flow out per unit time is given by

$$\dot{N} = \dot{V}\frac{p}{kT} = -\frac{\pi r^4}{8\eta}\frac{p}{kT}\frac{dp}{dz}. \qquad (5.5)$$

The mean number of the gas-molecules which pass through the unit area of the cross

section of the tube per unit time is given by

$$\Gamma = \frac{\dot{N}}{\pi r^2} = -\frac{r^2}{8\eta kT} p \frac{dp}{dz} = -\frac{r^2}{8\eta} p \frac{dn}{dz} = -\frac{r^2 nkT}{8\eta} \frac{dn}{dz}. \qquad \cdots\cdots (5.6)$$

Thus in this case, we can take, as the value of C, the following value,

$$C = \frac{r^2}{8\eta} p = \frac{r^2 kT}{8\eta} n. \qquad \cdots\cdots (5.7)$$

We compare the value of C with the value of D. Considering the relation $\eta = \frac{m\bar{v}}{3\pi d^2}$, the value of D can be expressed in the following form,

$$D = \frac{\bar{v}}{3\pi d^2 n} = \frac{\eta}{mn} = \frac{\eta}{\rho}. \qquad \cdots\cdots (5.8)$$

The ratio $\frac{C}{D}$ is given by

$$\frac{C}{D} = \frac{r^2 p\rho}{8\eta^2}. \qquad \cdots\cdots (5.9)$$

As an example, if we put $p = 10^4 \frac{dyne}{cm^2}$, $\rho = 0.0001 \frac{g}{cm^3}$ and $\eta = 0.00013\, C.G.S.$, we get as the ratio C/D:

$$\frac{C}{D} \doteqdot \frac{r^2 \times 10^7}{1.35}. \qquad \cdots\cdots (5.9')$$

Therefore, we can consider $C \gg D$ in usual cases. Thus if n_2 is not so small compared with n_1, we can put

$$\frac{dn_2}{dz} = -\frac{dn_1}{dz} \quad \text{or} \quad \frac{dn}{dz} = 0 \quad \text{or} \quad n = const. \qquad \cdots\cdots (5.10)$$

and

$$\Gamma_1' = -D \frac{n}{n_2} \frac{dn_1}{dz} = -\frac{A}{n_2} \frac{dn_1}{dz}. \qquad \cdots\cdots (5.4')$$

6. Numerical Values of Γ_1'

We calculate the numerical values of Γ_1' in a few simple cases. For simplicity, it is assumed that

$$\Gamma_1' = -D \frac{n}{n_2} \frac{dn_1}{dz} = -\frac{A}{n_2} \frac{dn_1}{dz} \qquad \cdots\cdots (5.4')$$

and

$$\frac{dn_2}{dz} = -\frac{dn_1}{dz} \qquad \cdots\cdots (5.10)$$

are applicable.

In the case of the stationary flow, as Γ_1' or $\frac{1}{n_2}\frac{dn_1}{dz}$ is constant, if we put

$$K=-\frac{1}{n_2}\frac{dn_1}{dz}=\frac{1}{n_2}\frac{dn_2}{dz}, \qquad (6.1)$$

then the value of K can be obtained, as follows, when the saturated vapor pressures at T_1^0K and T_2^0K, p_{1i} and p_{1f}, the total pressure of the mixture of the gas and the vapor, p, in the tube and the length of the tube l are given.

If the molecular densities of the vapor and the gas at T_1^0K end are expressed by n_{1i} and n_{2i}, and those at T_2^0K end are expressed by n_{1f} and n_{2f} respectively, and the pressures of the gas at T_1^0K end and T_2^0K end are expressed by p_{2i} and p_{2f} respectively, then we get

$$K=\frac{\ln n_{2f}-\ln n_{2i}}{l}=\frac{\ln p_{2f}-\ln p_{2i}}{l}, \qquad (6.1')$$

where

$$p_{2f}=p-p_{1f},\quad p_{2i}=p-p_{1i}.$$

Therefore, we can obtain the numerical value of Γ_1' by the following expression

$$\Gamma_1'=AK=\frac{\eta}{m}\cdot\frac{\ln p_{2f}-\ln p_{2i}}{l}. \qquad (6.2)$$

From (6.2) we calculate the values of Γ_1' for the water vapor by giving various values to T_1^0K, T_2^0K and p and by putting $l=100\ cm$. In these cases we take the values

$$\eta=1.28\times10^{-4}C.G.S.\quad\text{and}\quad m\doteqdot3\times10^{-23}g.$$

Table 1 shows the calculated values of Γ_1' for the water vapor. These values of Γ_1' are considered to be generally consistent with the observed values.

7. Conclusions

The phenomenon of vaporisation is very complicated, and is intimately related to the temperature, the pressure and other conditions, but it is able to study it quantitatively, if we can take a special case where the conditions are extremely simplified. We have studied here the stationary flow of the vapor of the liquid in the U-tube. We studied the quantities of the flowing vapor in the case of the stationary flow which is obtained, in the existence of the othe gas, by maintaining the temperatures of two legs at different fixed values, and obtained some results. By the results we can understand that how the rate of vaporisation is related to the temperature difference, to the pressure gradient of the vapor, to the pressure of the other gas in the tube, and to the sectional area and the form of the tube. As an

example, we calculated the numerical values of the rate of vaporisation of the water vapor under the various conditions. As to the numerical values about the vaporisation of the other liquids, we will study in the future.

References

(1) For example: M. W. Zemansky: Heat and Thermodynamics (1951) p. 282.
(2) For example: J. E. Mayer and M. G. Mayer: Statistical Mechanics (1940) p. 29.

Table 1. Values of Γ_1' ($10^{15}\mathrm{sec}^{-1}\mathrm{cm}^{-2}$) of water vapor in air
p_1……pressure of water vapor, p_2……pressure of air

(a) $p=p_1+p_2=760\mathrm{mmHg}$

t_1^0 C (T_1^0 K)	p_{1i} (mmHg)	t_2^0 C (T_2^0 K) →	0	10	20	30	40	50	60	70	80	90
		p_{1f} (mmHg)	4.6	9.2	17.5	31.8	55.3	92.5	149.4	233.7	355.1	525.8
		p_{2f} / p_{2i}	755.4	750.8	742.5	728.2	704.7	667.5	610.6	526.3	404.9	234.2
10	9.2	750.8	0.3									
20	17.5	742.5	0.8	0.5								
30	31.8	728.2	1.6	1.5	0.8							
40	55.3	704.7	3.0	2.8	2.3	1.4						
50	92.5	667.5	5.4	5.4	4.6	3.8	2.4					
60	149.4	610.6	9.2	9.0	8.5	7.6	6.2	3.9				
70	233.7	526.3	15.7	15.4	14.9	14.1	12.7	10.3	6.5			
80	355.1	404.9	27.1	26.8	26.3	25.5	24.1	21.7	17.8	11.1		
90	525.8	234.2	50.9	50.6	50.1	49.3	47.8	45.5	41.6	35.2	23.8	
95	633.9	126.1	77.7	77.5	77.0	76.1	74.7	72.4	68.5	62.1	50.7	26.9

Numerical values in [stepped box] are the values of Γ_1' ($10^{15}\mathrm{sec}^{-1}\mathrm{cm}^{-2}$)

(b) $p=p_1+p_2=600\mathrm{mmHg}$

t_1^0 C	p_{1i}	t_2^0 C →	0	10	20	30	40	50	60	70	80
		p_{1f}	4.6	9.2	17.5	31.8	55.3	92.5	149.4	233.7	355.1
		p_{2f} / p_{2i}	595.4	590.8	582.5	568.2	544.7	507.5	450.6	366.3	244.9
10	9.2	590.8	0.3								
20	17.5	582.5	0.9	0.6							
30	31.8	568.2	2.0	1.7	1.1						
40	55.3	544.7	3.9	3.5	2.9	1.8					
50	92.5	507.5	6.9	6.6	6.0	4.9	3.1				
60	149.4	450.6	12.4	11.8	11.1	10.1	8.2	5.2			
70	233.7	366.3	21.1	[illegible]	20.1	19.1	17.2	14.2	9.0		
80	355.1	244.9	38.6	38.2	37.6	36.6	[illegible]	31.0	[illegible]	[illegible]	
90	525.8	74.2	90.4	90.1	89.5	88.1	86.6	[illegible]	[illegible]	[illegible]	51.9

(c) $p=p_1+p_2=400\text{mmHg}$

		t_2^0 c	0	10	20	30	40	50	60	70
		p_{1f}	4.6	9.2	17.5	31.8	55.3	92.5	149.4	233.7
t_1^0 c	p_{1i}	p_{2i} \ p_{2f}	395.4	390.8	382.5	368.2	344.7	307.5	250.6	166.3
10	9.2	390.8	0.5							
20	17.5	382.5	1.4	0.9						
30	31.8	368.2	3.1	2.6	1.7					
40	55.3	344.7	6.0	5.5	4.5	2.9				
50	92.5	307.5	10.9	10.4	9.5	7.8	5.0			
60	149.4	250.6	19.8	19.3	18.4	16.7	13.8	8.9		
70	233.7	166.3	37.5	37.0	36.0	34.4	31.5	26.5	17.7	
80	355.1	44.9	94.5	94.0	93.0	91.4	88.5	83.6	74.7	57.0

(d) $p=p_1+p_2=250\text{mmHg}$

		t_2^0 C	0	10	20	30	40	50	60
		p_{1f}	4.6	9.2	17.5	31.8	55.3	92.5	149.4
t_1^0 C	p_{1i}	p_{2f} \ p_{2i}	245.4	240.8	232.5	218.2	194.7	157.5	100.6
10	9.2	240.8	0.8						
20	17.5	232.5	2.3	1.5					
30	31.8	218.2	5.1	4.3	2.8				
40	55.3	194.7	10.1	9.2	7.7	4.9			
50	92.5	157.5	19.3	18.4	16.9	14.2	9.2		
60	149.4	100.6	38.7	37.9	36.4	33.6	28.7	19.5	
70	233.7	16.3	117.8	116.9	115.4	112.7	107.7	98.5	79.0

THE STATIONARY FLOW IN MINAMATA BAY

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The object of this paper is to investigate the stationary stream line of the river water pouring into Minamata Bay, without taking the tidal flow into consideration. Of course, the actual stream line may be considerably blurred out by such tidal flow, and the results of calculation may be thought as only indicating the time average of the stream line. However, it can be said that it is no way useless to try such prediction, because, one of our purpose is to predict the distribution of the amount of deposit materials conveyed by river water from land, and it is easily supposed that only the average taken on considerably long time interval has any interesting relation to our problem. As seen on the added map Fig. 1, a general view of the bay is very

Fig. 1

similar to an ellipse whose ratio of major axis to minor is 2 to 1. The depth of the sea in the bay varies from 10m. to 20m.. The bay has on its east side two sources of water, which we may call q_1 and q_2 provisionally, namely, the river mouth at

Hyakken and a gush out water from the sea bottom in Fukuro area. Moreover, the bay has two narrow exits on the west side, which we replace by two sinks q_3 and q_4*. The former is a narrow channel between Koiji-shima and Yanagi-zaki, the latter is also between Koiji-shima and Myōjin-zaki. Naturally we can put

$$q_1 + q_2 + q_3 + q_4 = 0.$$

In this paper the program of computation will be as follows.

i) First we assume that the problem is two-dimentional.

ii) Next we solve the problem of the stationary flow in a circular basin with two sources and two sinks on the periphery using the well known image method in the theory of incompressible ideal fluid.

iii) Finally transform it to an elliptical basin by simple projection. Of course such projective transformation is not conformal, but the rough figure of distribution of the stream line will be not so seriously damaged by this process.

(1) *Stream Function*

If a source of intensity q lies on the periphery of the unit circle in the Argand-diagram, the stream function which represent the flow in the circle is given by

$$F(z) = \frac{q}{2\pi} \log \frac{(z-e^{i\alpha})^2}{z},$$

where α is the argument of q. This can be easily generalized to the case of many sources which can be described as,

$$F(z) = \sum_j \frac{q_j}{\pi} \log(z-e^{i\alpha_j}) - \sum_j \frac{q_j}{2\pi} \cdot \log z.$$

In our case, it is already said that

$$\sum_j q_j = 0,$$

and the stream funtion will be given by

$$F(z) = \frac{1}{\pi} \sum_j q_j \log(z-e^{i\alpha_j}),$$

where sign of q_j will be positive for a source and negative for a sink.

If φ and ψ be real and imaginary part of $F(z)$ respectively all stream lines are expressed by

$$\psi = const..$$

Replacing z by $re^{i\theta}$, then $F(z)$ is given by

$$F(z) = \frac{1}{2\pi} \sum_j q_j \log\{1+r^2-2r\cos(\theta-\alpha_j)\} + \frac{i}{\pi} \sum_j \tan^{-1}\left(\frac{r\sin\theta-\sin\alpha_j}{r\cos\theta-\cos\alpha_j}\right),$$

* q_3 and q_4 are naturally negative.

and hence, equation of stream lines become

$$\sum_j t_{an}^{-1} \frac{r\sin\theta - \sin\alpha_j}{r\cos\theta - \cos\alpha_j} = const..$$

(2) *Estimationof Constants*

The estimation of α_j was carried out in the following way. On the chart an ellipse whose ratio of major to minor axis is 2 to 1 was drawn so as to coincide with the shore line of the bay and the positions of the each sources and sinks were marked. Then the eccentric anomalies of these position will directly give the required angles in our circular model.

The following is a list of numbers obtained in this way.

q_1; $\sin\alpha_1 = -0.486$ $\cos\alpha_1 = 0.874$

q_2; $\sin\alpha_2 = -0.866$ $\cos\alpha_2 = -0.499$

q_3; $\sin\alpha_3 = 0.658$ $\cos\alpha_3 = -0.753$

q_4; $\sin\alpha_1 = 0.854$ $\cos\alpha_4 = 0.521.$

Estimation of q_j is more awkward than that of α_j, because the accurate survey of q_j have not yet been done up to present. We put $q_1 = 1$, for only the ratio of q_j is required. It may be thought that the ratio $q_3 : q_4$, i. e. the ratio of intensity of two sinks can be replaced by the ratio of cross sections of two exits of the bay. Rough number obtained from chart gives

$$q_3 : q_4 = 6 : 1.$$

If we put $q_2 = \gamma$ then all the q_j, are given by

$$q_1 = 1,$$

$$q_2 = \gamma,$$

$$q_3 = -\frac{6}{7}(1+\gamma),$$

$$q_4 = -\frac{1}{7}(1+\gamma).$$

Using q_j and α_j thus determined for several values of γ, numerical calculations have been carried out by the author, which are shown in the following tables.

The transformation to the elliptic basin from our model is immediately done by the multiplication of some number to x-components of all vectors and coordinates. The results of calculation are shown in Fig. 2 and Fig. 3.

Table 1

y \ x	−1.0	−0.9	−0.8	−0.7	−0.6	−0.5	−0.4	−0.3	−0.2	−0.1	0.0
1.0	—	—	—	—	—	—	—	—	—	—	8.91
0.9	—	—	—	—	—	—	8.86	8.76	8.72	8.72	8.75
0.8	—	—	—	—	8.91	8.60	8.47	8.44	8.46	8.50	8.57
0.7	—	—	—	8.66	8.07	7.99	8.02	8.09	8.17	8.27	8.38
0.6	—	—	3.53	6.06	6.96	7.31	7.53	7.70	7.87	8.04	8.17
0.4	—	3.62	4.30	5.07	5.75	6.28	6.69	7.01	7.29	7.54	7.76
0.2	—	3.86	4.33	4.83	5.31	5.76	6.15	6.50	6.82	7.11	7.38
0.0	3.53	3.88	4.25	4.65	5.04	5.43	5.80	6.15	6.47	6.77	7.05
−0.2	—	3.77	4.11	4.46	4.82	5.18	5.55	5.89	6.20	6.50	6.77
−0.4	—	3.57	3.88	4.22	4.59	4.98	5.36	5.72	6.04	6.33	6.58
−0.6	—	—	3.52	3.87	4.29	4.79	5.27	5.70	6.03	6.28	6.49
−0.7	—	—	—	—	—	—	—	—	—	-	—
−0.8	—	—	—	—	3.52	4.61	5.72	6.09	6.29	6.43	6.54
−0.9	—	—	—	—	—	—	6.54	6.47	6.50	6.55	6.60
−1.0	—	—	—	—	—	—	—	—	—	—	6.67

y \ x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	—	—	—	—	—	—	—	—	—	—
0.9	8.79	8.83	8.89	8.92	—	—	—	—	—	—
0.8	8.65	8.74	8.85	9.00	9.32	9.81	—	—	—	—
0.7	8.50	8.63	8.79	8.99	9.26	9.55	9.78	—	—	—
0.6	8.33	8.50	8.69	8.90	9.14	9.40	9.61	9.81	—	—
0.4	7.98	8.20	8.42	8.65	8.88	9.12	9.34	9.57	9.77	—
0.2	7.63	7.88	8.13	8.37	8.62	8.87	9.12	9.38	9.62	—
0.0	7.31	7.57	7.82	8.08	8.54	8.62	8.91	9.21	9.52	9.81
−0.2	7.03	7.28	7.51	7.75	8.01	8.29	8.63	9.03	9.47	—
−0.4	6.80	7.01	7.20	7.38	7.56	7.76	8.03	8.53	9.63	—
−0.6	6.66	6.80	6.92	7.01	7.07	7.08	6.99	6.67	—	—
−0.7	—	—	6.81	6.86	6.87	6.83	6.69	—	—	—
−0.8	6.62	6.69	6.73	6.75	6.73	6.66	—	—	—	—
−0.9	6.64	6.67	6.69	6.68	—	—	—	—	—	—
−1.0	—	—	—	—	—	—	—	—	—	—

Table of $-\pi\psi$

for $\gamma=1$, $q_1=1$, $q_2=1$, $q_3=-1.714$, $q_4=-0.286$.

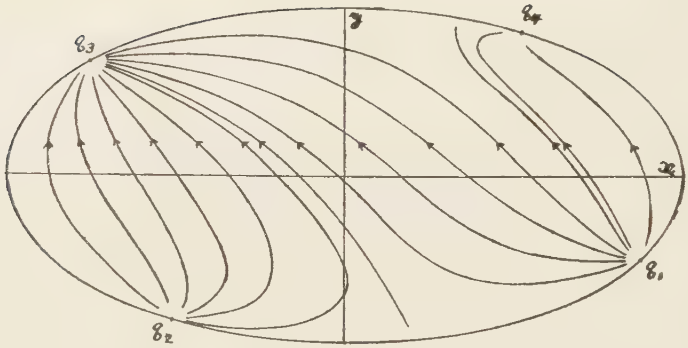
Table 2

y \ x	−1.0	−0.9	−0.8	−0.7	−0.6	−0.5	−0.4	−0.3	−0.2	−0.1	0.0
1.0	—	—	—	—	—	—	—	—	—	—	18.61
0.9	—	—	—	—	—	—	18.52	18.34	18.28	18.28	18.34
0.8	—	—	—	—	18.61	18.02	17.67	17.75	17.80	17.89	18.03
0.7	—	—	—	18.11	16.97	16.85	16.92	17.08	17.27	17.48	17.70
0.6	—	—	7.84	12.95	14.80	15.52	15.99	16.37	16.71	17.05	17.36
0.4	—	8.04	9.46	11.04	12.46	13.57	14.42	15.11	15.69	16.21	16.68
0.2	—	8.57	9.57	10.63	11.66	12.63	13.53	14.23	14.93	15.54	16.10
0.0	7.84	8.62	9.46	10.34	11.23	12.09	12.92	13.70	14.41	15.08	15.68
−0.2	—	8.42	9.19	10.02	10.88	11.73	12.59	13.40	14.14	14.83	15.45
−0.4	—	7.95	8.71	9.56	10.49	11.47	12.43	13.34	14.14	14.84	15.45
−0.6	—	—	7.84	8.73	9.87	11.23	12.53	13.72	14.58	15.24	15.76
−0.7	—	—	—	—	—	—	—	—	—	—	—
−0.8	—	—	—	—	7.83	11.00	14.23	15.26	15.78	16.15	16.43
−0.9	—	—	—	—	—	—	16.87	16.60	16.63	16.73	16.85
−1.0	—	—	—	—	—	—	—	—	—	—	17.27

y \ x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	—	—	—	—	—	—	—	—	—	—
0.9	18.41	18.49	18.60	18.64	—	—	—	—	—	—
0.8	18.19	18.37	18.58	18.84	19.46	20.40	—	—	—	—
0.7	17.94	18.20	18.50	18.88	19.39	19.94	20.36	—	—	—
0.6	17.68	18.01	18.37	18.77	19.21	19.68	20.07	20.41	—	—
0.4	17.15	17.56	17.98	18.41	18.83	19.25	19.64	20.01	20.35	—
0.2	16.62	17.12	17.60	18.05	18.49	18.94	19.34	19.74	20.12	—
0.0	16.24	16.77	17.26	17.73	18.20	18.66	19.11	19.55	20.00	20.41
−0.2	16.02	16.53	17.01	17.47	17.91	18.37	18.86	19.41	19.98	—
−0.4	15.98	16.44	16.86	17.24	17.60	17.96	18.36	18.99	20.21	—
−0.6	16.19	16.55	16.85	17.10	17.31	17.45	17.48	17.27	—	—
−0.7	—	—	16.92	17.10	17.24	17.31	17.28	—	—	—
−0.8	16.67	16.86	17.03	17.16	17.24	17.27	—	—	—	—
−0.9	16.97	17.08	17.18	17.25	—	—	—	—	—	—
−1.0	—	—	—	—	—	—	—	—	—	—

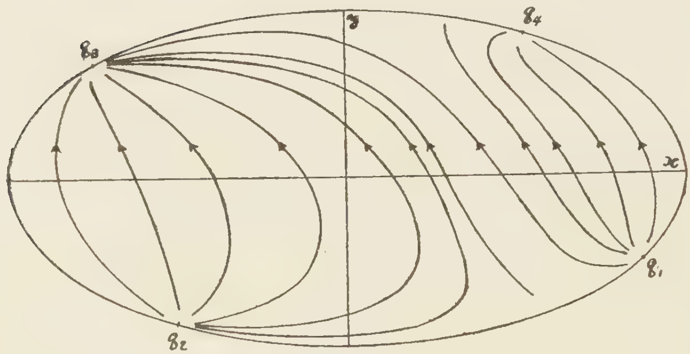
Table of $-\pi\psi$

for $\gamma=3$, $q_1=1$, $q_2=3$, $q_3=-3.429$, $q_4=-0.571$.



$\gamma = 1$

Fig. 2 Line of flow in the elliptical basin model.
($q_1=1$, $q_2=1$, $q_3=-1.714$, $q_4=-0.286$)



$\gamma = 3$

Fig. 3 Line of flow in the elliptical basin model
($q_1=1$, $q_2=3$, $q_3=-3.429$, $q_4=-0.571$

It is very suggestive that region inside of the ellipse separates distinctly to two regions, i. e. a region covered by those stream line which come from q_2 only, and other region come from q_1 only. Stream lines of latter do not enter into former region. Therefore, all kinds of river water effect in the q_2–region, of which source is considered to be q_1, are supposed solely come from blurring effect of sea tide, and it must be an interessant problem to study such dliffusional phenomena caused by the tidal flow.

NUMERICAL INTEGRATIONS FOR THE RADIATIVE CORES OF THE M DWARF STARS

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Abstract

Under the assumptions that the energy production is due to the p–p reaction ($\varepsilon \propto \rho T^5$) and the opacity is due to the photoionization of metals (modified Kramers' opacity $\kappa \propto \rho^{0.5} T^{-3.5}$), the numerical integrations for the cores of the M dwarf stars are computed for the three model series with the values of $\psi_c = 1.00$, 0.00 and $-\infty$, where ψ_c indicates the degree of degeneracy at the centre of the stars.

1. Introduction

At present, models having convective envelopes and radiative cores appear to describe the main-sequence stars later than the sun quite satisfactorily. In 1953 D. E. Osterbrock[1] applied these models to the K and the M_0 dwarf stars and obtained excellent agreements between the theory and the observation. Now we attempt to apply these models to the later M dwarf stars. In these stars the effect of the electron degeneracy on the pressure is not negligible, hence the equation of state becomes rather complicated. Fortunatelly, however, in the convective envelopes all the results of envelope solutions by Osterbrock[2] areavailable in their original forms for the present purpose, because *the degree of the electron degeneracy is constant throughout the convective region*[3]. Then it is enough to perform only the numerical integrations for the radiative cores.

2. Equations and Conditions

The four basic equilibrium conditions give the four differential equations

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2}, \qquad (1)$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho, \qquad (2)$$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \varepsilon \qquad (3)$$

and

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa \rho L_r}{T^3 4\pi r^2}, \qquad (4)$$

where the notations used are as uaual.

The three gas characteristic relations are

$$\left.\begin{aligned} P &= \frac{4\pi(2m)^{3/2}k^{5/2}}{h^3}\, T^{5/2}\Phi(\psi) \\ \rho &= \frac{4\pi H(2mk)^{3/2}}{h^3}\, T^{3/2}F_{1/2}(\psi) \end{aligned}\right\} \quad \text{(equation of state),} \quad \cdots\cdots\cdots\cdots\cdots \quad (5)$$

$$\kappa=\kappa_0\rho^{0.5}T^{-3.5} \qquad \text{(modified Kramers' opacity)} \quad \cdots\cdots\cdots\cdots\cdots \quad (6)$$

with $\kappa_0=1.81\times10^{25}Z(1+X)^{0.5}$

and

$$\varepsilon=\varepsilon_0\rho T^6 \qquad \text{(proton-proton reaction)} \quad \cdots\cdots\cdots\cdots\cdots \quad (7)$$

with $\varepsilon_0=\dfrac{0.0024X^2}{(5\times10^6)^6}$,

where $\Phi(\psi)=\dfrac{2}{3}F_{3/2}(\psi)+\dfrac{\mu_e}{\mu_i}F_{1/2}(\psi)$, $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (8)

H = the mass of the hydrogen atom,

$\dfrac{1}{\mu_e}$ = the free electron number per unit atomic weight,

$\dfrac{1}{\mu_i}$ = the ion number per unit atomic weight,

and

$$F_l(\psi)=\int_0^\infty \frac{x^l dx}{e^{x-\psi}+1}.$$

The boundary conditions are

$$L_r=0, \quad M_r=0 \quad \text{at} \quad r=0. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (9)$$

Now let us replace the physical variables M_r, L_r, r, P, T and ρ by the non-dimensional variables q, f, x, p, t and $\dfrac{p}{tg}$ with the help of the following standard transformations

$$M_r=qM,\ L_r=fL,\ r=xR,$$

$$P=p\,\frac{GM^2}{4\pi R^4},\quad T=t\,\frac{\mu H}{k}\,\frac{GM}{R}$$

and hence

$$\rho=\frac{p}{tg}\,\frac{M}{4\pi R^3}.$$

where

$$\frac{1}{g}=\frac{\mu_e}{\mu}\cdot\frac{F_{1/2}(\psi)}{\Phi(\psi)}\,. \qquad \cdots\cdots \qquad (10)$$

Then the eqs. (1) to (5) turn into the following forms;

$$\frac{dp}{dx}=-\frac{pq}{tx^2g} \qquad \cdots\cdots \qquad (11)$$

$$\frac{dq}{dx}=\frac{px^2}{tg} \qquad \cdots\cdots \qquad (12)$$

$$\frac{dt}{dx}=-C\frac{p^{1.5}f}{t^8x^2g^{1.5}} \qquad \cdots\cdots \qquad (13)$$

$$\frac{df}{dx}=Dp^2t^4x^2g^{-2} \qquad \cdots\cdots \qquad (14)$$

$$p=Ft^{2.5}\Phi(\psi) \qquad \cdots\cdots \qquad (15)$$

with

$$C=\frac{3\kappa_0}{4ac}\left(\frac{1}{4\pi}\right)^{2.5}\left(\frac{k}{\mu HG}\right)^{7.5}\frac{LR}{M^6}$$

$$D=\frac{\varepsilon_0}{4\pi}\left(\frac{\mu HG}{k}\right)^6\frac{M^6}{LR^9}$$

and

$$F=(4\pi)^2\left(\frac{2mG}{h^2}\right)^{1.5}(\mu H)^{2.5}(R^3M)^{1/2}.$$

If we apply the supplementary transformations

$$q=q_0q^*,\quad f=f_0f^*,\quad x=x_0x^*,\quad p=p_0p^*$$

and

$$t=t_0t^*$$

to the equations (11) to (15), we obtain

$$\frac{dp^*}{dx^*}=-\frac{p^*q^*}{t^*x^{*2}g} \qquad \cdots\cdots \qquad (16)$$

$$\frac{dq^*}{dx^*}=\frac{p^*x^{*2}}{t^*g} \qquad \cdots\cdots \qquad (17)$$

$$\frac{dt^*}{dx^*}=-\frac{p^{*1.5}f^*}{t^{*8}x^{*2}g^{1.5}} \qquad \cdots\cdots \qquad (18)$$

$$\frac{df^*}{dx^*}=p^{*2}t^{*4}x^{*2}g^{-2} \qquad \cdots\cdots \qquad (19)$$

$$p^* = t^{*2.5} I\Phi(\psi) \qquad (20)$$

with

$$\frac{q_0}{t_0 x_0} = 1,\ \frac{p_0 x_0^3}{t_0 q_0} = 1,\ C\frac{p_0^{1.5} f_0}{t_0^9 x_0} = 1,\ D\frac{p_0^2 t_0^4 x_0^3}{f_0} = 1,\ \frac{F t_0^{5/2}}{p_0} = I$$

and

$$t = t_c, \qquad (21)$$

where t_c designates the value of t at the center⁺.

Then the boundary conditions change into the following forms

$$q^* = f^* = 0 \quad \text{and} \quad t^* = 1 \quad \text{at} \quad x^* = 0. \qquad (22)$$

Finally if we introduce *the logarithmic variables:*

$$\left.\begin{array}{llll} \lambda = log\, p^* & \sigma = log\, q^* & \tau = log\, t^* & \\ \varphi = log\, f^* & y = log\, x^* & \gamma = log\, g & \delta = log\, \Phi(\psi), \end{array}\right\} \qquad (23)$$

we obtain *the logarithmic equations* as follows;

$$log\left(-\frac{d\lambda}{dy}\right) = \sigma - \tau - y - \gamma \qquad (24)$$

$$log\left(\frac{d\sigma}{dy}\right) = \lambda + 3y - \tau - \sigma - \gamma \qquad (25)$$

$$log\left(-\frac{d\tau}{dy}\right) = 1.5\lambda - 9\tau + \varphi - y - 1.5\gamma \qquad (26)$$

$$log\left(\frac{d\varphi}{dy}\right) = 2\lambda + 4\tau - \varphi + 3y - 2\gamma \qquad (27)$$

$$\delta = \lambda - 2.5\tau - \omega, \qquad (28)$$

where $\omega = logI$ is constant for each member of the integration courses. And we get the following four homology invariants:

$$U = \frac{d\sigma}{dy},\ V = -\frac{d\lambda}{dy},\ W = \frac{d\varphi}{dy} \qquad (29)$$

and

$$(n+1) = \frac{dlogP}{dlogT} = \frac{d\lambda/dy}{d\tau/dy}. \qquad (30)$$

The eqs. (24) to (28) and the conditions (22) show that we must further impose the two conditions in order to start the numerical integration. Thus the solutions of eqs. (24) to (28) form the *two-parameter series*, which may be classified by ψ_c and λ_c or alternatively by ψ_c and the value of the *polytropic index* at $x^* = 0$;

⁺ In this paper the suffix c refers to the central value.

$$log\,(n+1)_c = 1.5\gamma_c - 2.5\lambda_c\,. \quad \cdots\cdots \quad (31)$$

This relation follows from the equation (30), using the eqs. (24) and (26), and the condition $q^*/f^* = dq^*/df^*$ at $x^* = 0$.

The *starting values* for the integrations were found by power-series expansions valid near center ($x^* = 0$), using the eqs. (16) to (20) and the conditions (22);

$$q^* = \frac{1}{3}\frac{p_c^*}{g_c}x^{*3} - \frac{p_c^*}{30g_c^2}\left\{\frac{p_c^*}{g_c}\left[1 - \frac{p_c^{*2.5}}{g_c^{1.5}}\right] + 3\left(\frac{dg}{d\psi}\right)_c\psi_c''\right\}x^{*5}$$

$$+\left\{\frac{p_c^{*3}}{2520}\left[8 - 31.5\frac{p_c^{*2.5}}{g_c^{1.5}} + 38\frac{p_c^{*5}}{g_c^3}\right] + \frac{1}{1680}\left(\frac{dg}{d\psi}\right)_c\frac{p_c^{*2}}{g_c^1}\left[36 - 47\frac{p_c^{*2.5}}{g_c^{1.5}}\right]\psi_c''\right.$$

$$\left.+\frac{1}{168}\left(\frac{p_c^*}{g_c^2}\right)\left[\frac{6}{g_c}\left(\frac{dg}{d\psi}\right)^2(\psi_c'')^2 - 3\left(\frac{d^2g}{d\psi^2}\right)_c(\psi_c'')^2 - \left(\frac{dg}{d\psi}\right)_c\psi_c^{IV}\right]\right\}x^{*7}$$

$$t^* = 1 - \frac{1}{6}\left(\frac{p_c^*}{g_c}\right)^2 x^{*2} + \frac{p_c^*}{720g_c^2}\left\{\frac{p_c^*}{g_c}\left[27 - 56p_c^*\left(\frac{p_c^*}{g_c}\right)^{1.5}\right] + 81\left(\frac{dg}{d\psi}\right)_c\psi_c''\right\}x^{*4}$$

$$p^* = p_c^* - \frac{1}{6}\left(\frac{p_c^*}{g_c}\right)^2 x^{*2} + \frac{1}{45}\frac{p_c^*}{g_c^2}\left\{\frac{p_c^*}{g_c}\left[1 - p_c^*\left(\frac{p_c^*}{g_c}\right)^{1.5}\right] + 3\left(\frac{dg}{d\psi}\right)_c\psi_c''\right\}x^{*4}$$

$$f^* = \frac{1}{3}\left(\frac{p_c^*}{g_c}\right)^2 x^{*3} - \frac{p_c^{*2}}{15g_c^2}\left\{\frac{p_c^*}{g_c}\left[1 + 2\frac{p_c^{*2.5}}{g_c^{1.5}}\right] + 3\left(\frac{dg}{d\psi}\right)_c\psi_c''\right\}x^{*5}$$

$$+\frac{p_c^{*2}}{84g_c^2}\left\{\frac{p_c^{*2}}{15g_c^2}\left[13 + 59\frac{p_c^{*2.5}}{g_c^{1.5}} - 46\frac{p_c^{*5}}{g_c^3}\right] + \frac{1}{2}\left\{\left(\frac{dg}{d\psi}\right)_c\psi_c''\left[11.2 + 26.8\frac{p_c^{*2.5}}{g_c^{1.5}}\right]\left(\frac{p_c^*}{g_c^2}\right)\right.\right.$$

$$\left.\left.+\frac{1}{g_c}\left[\left(\frac{9}{g_c}\left(\frac{dg}{d\psi}\right)_c^2(\psi_c'')^2 - 3(\psi_c'')^2\left(\frac{d^2g}{d\psi^2}\right)_c - \left(\frac{dg}{d\psi}\right)_c\psi_c^{IV}\right)\right]\right\}\right\}x^{*7}$$

where

$$\psi_c'' = \left(\frac{d^2\psi}{dx^{*2}}\right)_c = \frac{1}{3}\left(\frac{d\psi}{d\Phi}\right)_c\left(\frac{p_c^*}{g_c}\right)\left[\frac{5}{2}\left(\frac{p_c^*}{g_c}\right)^{1.5} - \frac{\Phi_c}{p_c^*}\right]$$

$$\psi_c^{IV} = \left(\frac{d^4\psi}{dx_c^{*4}}\right)_c = \frac{\Phi_c p_c^*}{60g_c^3}\left(\frac{d\psi}{d\Phi}\right)_c\left\{\frac{p_c^*}{g_c}\left[32 - 167\frac{p_c^{*2.5}}{g_c^{1.5}} + 205\frac{p_c^{*5}}{g_c^3}\right]\right.$$

$$\left.+\left(\frac{dg}{d\psi}\right)_c\psi_c''\left[94 - 405\frac{p_c^{*2.5}}{g_c^{1.5}}\right]\right\} + 5\left(\frac{p_c^*}{g_c}\right)^{3.5}\psi_c'' - 3\left(\frac{d\psi}{d\Phi}\right)_c\left(\frac{d^2\Phi}{d\psi^2}\right)_c(\psi_c'')$$

The *parameter* $(n+1)_c = g_c^{1.5}p_c^{*-2.5}$ is listed for each series in the Table 1.

Table 1

Series 1 (ψ_c=1.00)		Series 2 (ψ_c=0.00)		Series 3 ($\psi_c=-\infty$)	
No.	$(n+1)_c$	No.	$(n+1)_c$	No.	$(n+1)_c$
1.1	2.80	2.1	2.80	3.1	2.80
1.2	2.85	2.2	2.82	3.2	2.81
1.3	2.86	2.3	2.83	3.3	2.82
1.4	2.861	2.4	2.84	3.4	2.822
1.5	2.861 5	2.5	2.8425	3.5	2.825
1.6	2.861 6	2.6	2.84325	3.6	2.83
1.7	2.861 677	2.7	2.85	3.7	2.84
1.8	2.861 7				
1.9	2.862				
1.10	2.87				
1.11	2.89				

Only integrations 1.1, 1.2, 1.11, 2.1, 2.7, 3.1, and 3.7 are directly started near the center with their starting values. All other integrations are started, successively, farther out, by interpolation between neighboring integrations.

3. Numerical Methods

All calculations were carried out by the aid of the desk calculating machine. At first the functions $\gamma(\psi)$ and $\delta(\psi)$ have been obtained from the calculations that have been carried out by Mc Dougall and Stoner(4), and is given in the Table 2 for the case :

$$X=0.80,\quad Y=0.18,\quad Z=0.02.^{++}$$

From the Table 2 the value of γ is determined by the value of δ which is calculated by eq. (28) from the values of τ and λ. (In the case of the perfect gas models $\psi=-\infty$, there is no necessity for calculating the values of γ and δ.)

The integration scheme used is given by

$$x(z+\triangle)=x(z)+\triangle x'(z)+\frac{1}{2}\,{}^{1}D\left(z+\frac{1}{2}\triangle\right)-\frac{1}{12}\,{}^{2}D(z)-\frac{1}{24}\,{}^{3}D\left(z-\frac{1}{2}\triangle\right).$$

Here x stands for any of the dependent variables except γ and δ and x' stands for its derivative given by the differential equations. The D's stand for the successive differences of $\triangle\cdot x'$. The last three terms in this equation are initially obtained at every step from a forward guess. This guess is checked by computing $\triangle\cdot x'(z+\triangle)$

++ In the present paper all actual numerical calculations have been carried out for this composition. However, since both values γ and δ do not seriously depend upon the composition, but mainly depend on the value of ψ, it seems quite sufficient for our purpose to carry out the numerical calculation only for this composition.

Table 2

The Functions $\delta(\psi)$ and $\gamma(\psi)$ ($x=0.80$, $z=0.02$)

ψ	$\delta(\psi)$	$\gamma(\psi)$	ψ	$\delta(\psi)$	$\gamma(\psi)$
1.0	+0.489 32	0.056 79	−2.5	−0.859 74	0.003 16
0.9	+0.457 39	0.053 46	−2.6	−0.902 32	0.002 86
0.8	+0.424 94	0.050 23	−2.7	−0.944 98	0.002 60
0.7	+0.392 00	0.047 16	−2.8	−0.987 72	0.002 34
0.6	+0.358 55	0.044 19	−2.9	−1.030 51	0.002 12
0.5	+0.324 60	0.041 31	−3.0	−1.073 36	0.001 91
0.4	+0.290 17	0.038 58	−3.1	−1.116 27	0.001 73
0.3	+0.255 26	0.035 99	−3.2	−1.159 22	0.001 60
0.2	+0.219 87	0.033 50	−3.3	−1.202 23	0.001 43
0.1	+0.184 20	0.031 13	−3.4	−1.245 26	0.001 30
0.0	+0.147 73	0.028 90	−3.5	−1.288 34	0.001 17
−0.1	+0.111 01	0.026 78	−3.6	−1.331 46	0.001 08
−0.2	+0.073 86	0.024 77	−3.7	−1.374 58	0.001 00
−0.3	+0.036 30	0.022 88	−3.8	−1.417 75	0.000 91
−0.4	−0.001 65	0.021 11	−3.9	−1.460 95	0.000 78
−0.5	−0.039 97	0.019 49	−4.0	−1.504 16	0.000 74
−0.6	−0.078 65	0.017 91	−4.1	−1.547 39	0.000 65
−0.7	−0.117 66	0.016 45	−4.2	−1.590 64	0.000 61
−0.8	−0.157 01	0.015 11	−4.3	−1.633 91	0.000 52
−0.9	−0.196 65	0.013 85	−4.4	−1.677 20	0.000 48
−1.0	−0.236 59	0.012 71	−4.5	−1.720 49	0.000 43
−1.1	−0.276 79	0.011 66	−4.6	−1.763 79	0.000 39
−1.2	−0.317 25	0.010 64	−4.7	−1.807 15	0.000 36
−1.3	−0.357 95	0.009 71	−4.8	−1.850 44	0.000 33
−1.4	−0.398 87	0.008 90	−4.9	−1.893 78	0.000 30
−1.5	−0.439 99	0.008 09	−5.0	−1.937 15	0.000 27
−1.6	−0.481 31	0.007 41	−5.1	−1.980 51	0.000 24
−1.7	−0.522 81	0.006 72	−5.2	−2.023 79	0.000 21
−1.8	−0.564 47	0.006 12	−5.3	−2.067 22	0.000 19
−1.9	−0.606 28	0.005 57	−5.4	−2.110 53	0.000 17
−2.0	−0.648 23	0.005 10	−5.5	−2.153 97	0.000 16
−2.1	−0.690 31	0.004 62	−5.6	−2.197 36	0.000 15
−2.2	−0.732 51	0.004 19	−5.7	−2.240 71	0.000 14
−2.3	−0.774 82	0.003 81	−5.8	−2.284 08	0.000 13
−2.4	−0.817 23	0.003 46	−5.9	−2.327 62	0.000 11
−2.5	−0.85974	0.003 16	−6.0	−2.370 90	0.000 10

with the help of the differential equation, using the values of $\varkappa(z+\triangle)$ and $\gamma(z+\triangle)$ obtained from the foregoing equations. Then the necessary D-values can be computed. If the computed values do not agree with the guessed values within two at the last digit, the cycle is repeated+++. Thus we can carry the numerical integrations step by step outward from the center to the point at which $(n+1)$ falls to 2.5. Outward from this point a convective envelope must be fitted so as to construct a complete stellar model.

All integration steps executed are reproduced in the following tabulation. The integrations were performed with one more digit than are given in the tabulation.

+++ The step length, $\triangle$, is so chosen that the fourth difference have no influence on the integration formula and that forward guessing of the lower differences is reasonably easy.

4. Tabulation of Integrations

In the following tabulation each integration is characterized by a number which consists of the series number in front of the decimal point and the individual integration number after the decimal point. The column headings have the following meanings;

(1) $y = \log x^* = \log \frac{x}{x_0}$	(5) $\varphi = \log f^* = \log \frac{f}{f_0}$	(9) $n+1 = \frac{d\lambda}{d\tau}$
(2) $\sigma = \log q^* = \log \frac{q}{q_0}$	(6) $U = \frac{d\sigma}{dy}$	(10) ψ
(3) $\tau = \log t^* = \log \frac{t}{t_c}$	(7) $V = -\frac{d\lambda}{dy}$	(11) δ
(4) $\lambda = \log p^* = \log \frac{p}{p_0}$	(8) $W = \frac{d\varphi}{dy}$	(12) γ

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1. 1

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
−0.62	−2.539 5	−0.000 82	−0.147 1	−2.714 8	2.996	0.010 58	2.977	2.805	0.999	0.489 1	0.056 8
−0.60	−2.479 6	−0.000 90	−0.147 3	−2.685 3	2.995	0.011 60	2.975	2.805	0.999	0.489 0	0.056 8
−0.58	−2.419 7	−0.000 99	−0.147 6	−2.625 8	2.995	0.012 72	2.972	2.805	0.999	0.489 0	0.056 8
−0.56	−2.359 8	−0.001 08	−0.147 8	−2.566 4	2.994	0.013 95	2.970	2.806	0.999	0.489 0	0.056 8
−0.54	−2.299 9	−0.001 19	−0.148 1	−2.507 0	2.993	0.015 29	2.968	2.807	0.999	0.489 0	0.056 8
−0.52	−2.240 0	−0.001 30	−0.148 5	−2.447 7	2.993	0.016 77	2.965	2.808	0.999	0.488 9	0.056 8
−0.50	−2.180 2	−0.001 42	−0.148 8	−2.388 4	2.992	0.018 38	2.962	2.808	0.999	0.488 9	0.056 8
−0.48	−2.120 4	−0.001 56	−0.149 2	−2.329 3	2.992	0.020 16	2.958	2.809	0.998	0.488 8	0.056 8
−0.46	−2.060 5	−0.001 71	−0.149 6	−2.270 1	2.991	0.022 10	2.954	2.810	0.998	0.488 8	0.056 8
−0.44	−2.000 7	−0.001 88	−0.150 1	−2.211 1	2.990	0.024 23	2.950	2.811	0.998	0.488 7	0.056 8
−0.42	−1.940 9	−0.002 06	−0.150 6	−2.152 1	2.990	0.026 57	2.946	2.812	0.998	0.488 7	0.056 8
−0.40	−1.881 1	−0.002 25	−0.151 1	−2.093 3	2.989	0.028 13	2.940	2.813	0.998	0.488 6	0.056 8
−0.38	−1.821 1	−0.002 47	−0.151 8	−2.034 5	2.988	0.031 91	2.935	2.814	0.998	0.488 5	0.056 8
−0.36	−1.761 6	−0.002 71	−0.152 4	−1.975 9	2.987	0.035 02	2.930	2.815	0.997	0.488 5	0.056 7
−0.34	−1.701 9	−0.002 97	−0.153 2	−1.917 4	2.986	0.038 42	2.923	2.816	0.997	0.488 4	0.056 7
−0.32	−1.642 2	−0.003 26	−0.154 0	−1.859 0	2.984	0.042 11	2.915	2.818	0.997	0.488 3	0.056 7
−0.30	−1.582 6	−0.003 57	−0.154 8	−1.800 8	2.983	0.046 17	2.907	2.820	0.996	0.488 2	0.056 7
−0.28	−1.522 9	−0.003 91	−0.155 8	−1.742 7	2.981	0.050 62	2.898	2.822	0.996	0.488 1	0.056 7
−0.26	−1.463 3	−0.004 29	−0.156 9	−1.684 9	2.979	0.055 50	2.888	2.824	0.996	0.488 0	0.056 7
−0.24	−1.403 8	−0.004 70	−0.158 0	−1.627 2	2.977	0.060 85	2.877	2.827	0.995	0.487 8	0.056 7
−0.22	−1.344 3	−0.005 15	−0.159 3	−1.569 8	2.975	0.066 75	2.865	2.829	0.995	0.487 7	0.056 7
−0.20	−1.284 8	−0.005 65	−0.160 7	−1.512 6	2.972	0.073 11	2.852	2.832	0.994	0.487 5	0.056 7
−0.18	−1.225 4	−0.006 19	−0.162 2	−1.455 7	2.970	0.080 20	2.840	2.835	0.994	0.487 3	0.056 6
−0.16	−1.166 0	−0.006 78	−0.163 9	−1.399 1	2.967	0.087 94	2.824	2.838	0.993	0.487 1	0.056 6
−0.14	−1.106 7	−0.007 43	−0.165 8	−1.342 7	2.963	0.096 11	2.807	2.842	0.992	0.486 9	0.056 6
−0.12	−1.047 5	−0.008 11	−0.167 8	−1.286 8	2.960	0.105 7	2.789	2.846	0.992	0.486 7	0.056 6
−0.10	−0.988 3	−0.008 91	−0.170 0	−1.231 2	2.956	0.116 0	2.770	2.851	0.991	0.486 4	0.056 5
−0.08	−0.929 2	−0.009 76	−0.172 1	−1.176 0	2.952	0.127 1	2.748	2.855	0.990	0.486 1	0.056 5
−0.06	−0.870 3	−0.010 70	−0.175 1	−1.121 3	2.947	0.139 3	2.724	2.860	0.989	0.485 8	0.056 5
−0.04	−0.811 4	−0.011 71	−0.178 0	−1.067 0	2.942	0.152 7	2.700	2.865	0.988	0.485 4	0.056 4
−0.02	−0.752 6	−0.012 83	−0.181 2	−1.013 3	2.937	0.167 1	2.672	2.871	0.987	0.485 0	0.056 4
−0.00	−0.693 9	−0.014 05	−0.184 7	−0.960 2	2.930	0.183 5	2.641	2.878	0.985	0.484 5	0.056 4
+0.02	−0.635 4	−0.015 38	−0.188 6	−0.907 7	2.924	0.201 2	2.609	2.886	0.983	0.484 0	0.056 3
+0.04	−0.577 0	−0.016 84	−0.192 8	−0.855 9	2.917	0.220 6	2.573	2.893	0.982	0.483 4	0.056 2
+0.06	−0.518 8	−0.018 44	−0.197 4	−0.804 8	2.908	0.241 8	2.534	2.903	0.980	0.482 8	0.056 2
+0.08	−0.460 7	−0.020 18	−0.202 5	−0.754 5	2.899	0.265 1	2.492	2.912	0.977	0.482 1	0.056 1
+0.10	−0.402 8	−0.022 08	−0.208 0	−0.705 1	2.890	0.290 6	2.447	2.922	0.975	0.481 3	0.056 0

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.12	−0.315 1	−0.024 16	−0.214 1	−0.656 6	2.879	0.318 5	2.398	2.933	0.972	0.480 4	0.055 9
+0.14	−0.287 7	−0.026 43	−0.220 8	−0.609 2	2.868	0.349 1	2.316	2.946	0.969	0.479 4	0.055 8
+0.16	−0.230 4	−0.028 91	−0.228 1	−0.562 8	2.855	0.382 6	2.289	2.959	0.965	0.478 3	0.055 7
+0.18	−0.173 5	−0.031 61	−0.236 1	−0.517 7	2.840	0.419 3	2.227	2.974	0.961	0.477 0	0.055 6
+0.20	−0.116 8	−0.034 56	−0.244 9	−0.473 8	2.825	0.459 1	2.161	2.990	0.957	0.475 6	0.055 5
+0.22	−0.060 5	−0.037 77	−0.254 5	−0.431 2	2.809	0.503 4	2.092	3.006	0.952	0.474 0	0.055 3
+0.24	−0.004 5	−0.041 26	−0.265 1	−0.390 1	2.790	0.551 6	2.017	3.024	0.946	0.472 2	0.055 1
+0.26	+0.051 1	−0.045 07	−0.276 6	−0.350 6	2.770	0.601 3	1.938	3.044	0.940	0.470 2	0.054 9
+0.28	+0.106 3	−0.049 21	−0.289 3	−0.312 7	2.748	0.661 9	1.853	3.066	0.933	0.467 8	0.054 7
+0.30	+0.161 0	−0.053 72	−0.303 1	−0.276 5	2.724	0.721 9	1.764	3.088	0.925	0.465 3	0.054 4
+0.32	+0.215 2	−0.058 61	−0.318 3	−0.242 1	2.698	0.793 8	1.671	3.111	0.916	0.462 3	0.054 1
+0.34	+0.268 9	−0.063 93	−0.334 9	−0.209 7	2.670	0.869 2	1.573	3.136	0.905	0.459 0	0.053 7
+0.36	+0.322 0	−0.069 71	−0.353 1	−0.179 3	2.638	0.951 4	1.471	3.162	0.893	0.455 3	0.053 3
+0.38	+0.374 4	−0.075 98	−0.373 0	−0.150 9	2.604	1.041	1.365	3.188	0.880	0.451 0	0.052 9
+0.40	+0.426 1	−0.082 79	−0.394 8	−0.124 7	2.567	1.139	1.256	3.213	0.866	0.446 3	0.052 4
+0.42	+0.477 1	−0.090 18	−0.418 7	−0.100 7	2.527	1.246	1.146	3.238	0.849	0.440 9	0.051 8
+0.44	+0.527 2	−0.098 20	−0.444 7	−0.078 9	2.484	1.362	1.033	3.260	0.831	0.434 9	0.051 2
+0.46	+0.576 1	−0.106 91	−0.473 2	−0.059 3	2.437	1.489	0.921	3.277	0.810	0.428 2	0.050 5
+0.48	+0.624 6	−0.116 39	−0.504 3	−0.042 0	2.386	1.626	0.810	3.288	0.787	0.420 8	0.049 8
+0.50	+0.671 8	−0.126 73	−0.538 1	−0.027 0	2.331	1.776	0.701	3.288	0.762	0.412 6	0.049 1
+0.52	+0.717 8	−0.138 04	−0.575 5	−0.014 0	2.272	1.939	0.595	3.273	0.736	0.403 7	0.048 4
+0.54	+0.762 7	−0.150 49	−0.616 0	−0.003 1	2.210	2.117	0.495	3.231	0.707	0.394 3	0.047 4
+0.56	+0.806 2	−0.164 32	−0.660 3	+0.005 8	2.144	2.312	0.401	3.159	0.678	0.384 7	0.046 6
+0.57	+0.827 5	−0.171 87	−0.683 9	+0.009 6	2.111	2.417	0.357	3.103	0.664	0.379 9	0.046 1
+0.58	+0.848 1	−0.179 93	−0.708 6	+0.013 0	2.076	2.527	0.315	3.033	0.650	0.375 3	0.045 7
+0.59	+0.869 0	−0.188 58	−0.731 5	+0.015 9	2.041	2.644	0.276	2.943	0.637	0.371 1	0.045 3
+0.60	+0.889 2	−0.197 93	−0.761 5	+0.018 5	2.005	2.768	0.238	2.832	0.626	0.367 4	0.045 0
+0.605	+0.899 2	−0.202 93	−0.775 5	+0.019 6	1.988	2.833	0.220	2.766	0.622	0.365 9	0.044 9
+0.610	+0.909 1	−0.208 18	−0.789 9	+0.020 7	1.970	2.900	0.203	2.692	0.619	0.364 7	0.044 8
+0.615	+0.918 9	−0.213 72	−0.804 5	+0.021 7	1.953	2.971	0.186	2.610	0.616	0.363 9	0.044 7
+0.620	+0.928 6	−0.219 59	−0.819 6	+0.022 5	1.936	3.045	0.170	2.518	0.615	0.363 5	0.044 6
+0.625	+0.938 2	−0.225 81	−0.835 0	+0.023 4	1.918	3.122	0.155	2.416	0.615	0.363 7	0.044 7

[1. 2]

−0.62	−2.512 6	−0.000 80	−0.150 1	−2.751 0	2.995	0.010 51	2.977	2.855	0.999	0.489 1	0.056 8
−0.60	−2.482 7	−0.000 88	−0.150 1	−2.691 5	2.995	0.011 52	2.975	2.855	0.999	0.489 0	0.056 8

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
−0.58	−2.422 8	−0.000 96	−0.150 6	−2.632 0	2.994	0.012 63	2.972	2.855	0.999	0.489 0	0.056 8
−0.56	−2.362 9	−0.001 05	−0.150 9	−2.572 5	2.994	0.013 85	2.970	2.856	0.999	0.489 0	0.056 8
−0.54	−2.303 0	−0.001 16	−0.151 2	−2.513 1	2.994	0.015 18	2.968	2.857	0.999	0.488 9	0.056 8
−0.52	−2.243 1	−0.001 27	−0.151 5	−2.453 7	2.993	0.016 65	2.966	2.858	0.999	0.488 9	0.056 8
−0.50	−2.183 3	−0.001 39	−0.151 8	−2.394 5	2.993	0.018 25	2.962	2.859	0.998	0.488 8	0.056 8
−0.48	−2.123 5	−0.001 52	−0.152 2	−2.335 2	2.992	0.020 01	2.959	2.860	0.998	0.488 8	0.056 8
−0.46	−2.063 6	−0.001 67	−0.152 6	−2.276 1	2.991	0.021 91	2.955	2.861	0.998	0.488 7	0.056 8
−0.44	−2.003 8	−0.001 83	−0.153 1	−2.217 0	2.991	0.024 06	2.951	2.862	0.998	0.488 7	0.056 8
−0.42	−1.944 0	−0.002 01	−0.153 6	−2.158 1	2.990	0.026 38	2.946	2.863	0.998	0.488 6	0.056 8
−0.40	−1.884 2	−0.002 20	−0.154 2	−2.099 2	2.989	0.028 92	2.941	2.864	0.997	0.488 5	0.056 8
−0.38	−1.824 5	−0.002 41	−0.154 8	−2.040 4	2.988	0.031 71	2.937	2.865	0.997	0.488 4	0.056 7
−0.36	−1.764 7	−0.002 64	−0.155 4	−1.981 7	2.987	0.034 77	2.930	2.867	0.997	0.488 4	0.056 7
−0.34	−1.705 0	−0.002 90	−0.156 2	−1.923 2	2.986	0.038 13	2.924	2.868	0.997	0.488 3	0.056 7
−0.32	−1.645 3	−0.003 18	−0.157 0	−1.864 8	2.984	0.041 80	2.916	2.870	0.996	0.488 2	0.056 7
−0.30	−1.585 6	−0.003 48	−0.157 8	−1.806 6	2.982	0.045 83	2.908	2.872	0.996	0.488 1	0.056 7
−0.28	−1.526 0	−0.003 82	−0.158 8	−1.748 5	2.981	0.050 25	2.899	2.874	0.995	0.487 9	0.056 7
−0.26	−1.466 1	−0.004 18	−0.159 9	−1.690 6	2.979	0.055 09	2.889	2.877	0.995	0.487 8	0.056 7
−0.24	−1.406 9	−0.004 58	−0.161 0	−1.633 0	2.977	0.060 40	2.879	2.879	0.995	0.487 6	0.056 7
−0.22	−1.347 3	−0.005 02	−0.162 3	−1.575 5	2.975	0.066 24	2.869	2.882	0.994	0.487 5	0.056 6
−0.20	−1.287 9	−0.005 51	−0.163 7	−1.518 2	2.973	0.072 62	2.856	2.885	0.994	0.487 3	0.056 6
−0.18	−1.228 5	−0.006 03	−0.165 2	−1.461 3	2.970	0.079 62	2.842	2.888	0.993	0.487 1	0.056 6
−0.16	−1.169 1	−0.006 61	−0.166 8	−1.404 6	2.967	0.087 28	2.827	2.892	0.992	0.486 9	0.056 6
−0.14	−1.109 8	−0.007 24	−0.168 7	−1.348 2	2.963	0.095 69	2.810	2.896	0.992	0.486 6	0.056 6
−0.12	−1.050 6	−0.007 93	−0.170 7	−1.292 2	2.960	0.104 9	2.793	2.900	0.991	0.486 4	0.056 5
−0.10	−0.991 1	−0.008 69	−0.172 9	−1.236 5	2.956	0.115 0	2.773	2.905	0.990	0.486 0	0.056 5
−0.08	−0.932 3	−0.009 52	−0.175 3	−1.181 3	2.952	0.126 1	2.752	2.911	0.989	0.485 7	0.056 5
−0.06	−0.873 5	−0.010 43	−0.177 9	−1.126 4	2.947	0.138 3	2.730	2.916	0.988	0.485 3	0.056 4
−0.04	−0.814 4	−0.011 42	−0.180 8	−1.072 1	2.942	0.151 6	2.704	2.923	0.986	0.484 9	0.056 4
−0.02	−0.755 6	−0.012 50	−0.184 0	−1.018 3	2.937	0.166 2	2.677	2.930	0.985	0.484 4	0.056 3
0.00	−0.697 0	−0.013 69	−0.187 5	−0.965 0	2.930	0.182 2	2.647	2.937	0.983	0.483 9	0.056 3
+0.02	−0.638 1	−0.014 99	−0.191 3	−0.912 4	2.924	0.199 7	2.615	2.945	0.981	0.483 4	0.056 2
+0.04	−0.580 0	−0.016 41	−0.195 5	−0.860 5	2.916	0.218 9	2.579	2.955	0.979	0.482 7	0.056 2
+0.06	−0.521 8	−0.017 95	−0.200 1	−0.809 3	2.908	0.239 9	2.542	2.965	0.977	0.482 0	0.056 1
+0.08	−0.463 7	−0.019 65	−0.205 1	−0.758 8	2.899	0.263 0	2.501	2.976	0.975	0.481 2	0.056 0
+0.10	−0.405 9	−0.021 49	−0.210 6	−0.709 3	2.890	0.288 2	2.456	2.988	0.972	0.480 3	0.055 9
+0.12	−0.348 2	−0.023 51	−0.216 6	−0.660 6	2.879	0.315 9	2.408	3.001	0.969	0.479 3	0.055 8
+0.14	−0.290 7	−0.025 71	−0.223 3	−0.612 9	2.867	0.346 2	2.356	3.014	0.965	0.478 2	0.055 7
+0.16	−0.233 5	−0.028 11	−0.230 5	−0.566 4	2.854	0.379 3	2.300	3.031	0.961	0.477 0	0.055 6
+0.18	−0.176 5	−0.030 72	−0.238 4	−0.521 0	2.840	0.415 6	2.240	3.048	0.957	0.475 6	0.055 5
+0.20	−0.119 9	−0.033 57	−0.247 1	−0.476 8	2.825	0.455 4	2.176	3.067	0.952	0.474 0	0.055 3

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.22	−0.063 6	−0.036 67	−0.256 7	−0.434 0	2.808	0.498 9	2.107	3.088	0.946	0.472 2	0.055 1
+0.24	−0.007 6	−0.040 04	−0.267 1	−0.392 5	2.790	0.546 5	2.034	3.110	0.940	0.470 2	0.054 9
+0.26	+0.048 0	−0.043 70	−0.278 6	−0.352 6	2.770	0.598 6	1.957	3.135	0.933	0.467 9	0.054 6
+0.28	+0.103 2	−0.047 68	−0.291 1	−0.314 3	2.748	0.655 6	1.874	3.161	0.925	0.465 3	0.054 3
+0.30	+0.157 9	−0.052 01	−0.304 8	−0.277 7	2.721	0.717 7	1.787	3.190	0.916	0.462 4	0.054 0
+0.32	+0.212 2	−0.056 69	−0.319 8	−0.242 9	2.698	0.785 6	1.695	3.222	0.905	0.459 1	0.053 7
+0.34	+0.265 8	−0.061 77	−0.336 3	−0.209 9	2.669	0.859 6	1.599	3.256	0.891	0.455 3	0.053 3
+0.36	+0.318 9	−0.067 26	−0.354 2	−0.178 9	2.638	0.940 3	1.498	3.293	0.881	0.451 1	0.052 9
+0.38	+0.371 3	−0.073 20	−0.373 9	−0.150 0	2.604	1.028	1.395	3.333	0.866	0.446 3	0.052 4
+0.40	+0.423 0	−0.079 61	−0.395 1	−0.123 2	2.567	1.124	1.289	3.375	0.848	0.440 8	0.051 8
+0.42	+0.474 0	−0.086 53	−0.418 9	−0.098 5	2.526	1.228	1.181	3.421	0.830	0.434 6	0.051 2
+0.44	+0.524 0	−0.093 98	−0.444 6	−0.075 9	2.482	1.341	1.071	3.469	0.808	0.427 5	0.050 5
+0.46	+0.573 2	−0.102 01	−0.472 6	−0.055 6	2.438	1.461	0.962	3.521	0.784	0.419 6	0.049 7
+0.48	+0.621 1	−0.110 63	−0.503 2	−0.037 5	2.383	1.596	0.853	3.571	0.756	0.410 5	0.048 9
+0.50	+0.668 5	−0.119 89	−0.536 6	−0.021 5	2.328	1.740	0.717	3.628	0.725	0.400 4	0.047 9
+0.52	+0.714 5	−0.129 82	−0.572 9	−0.007 6	2.268	1.891	0.611	3.684	0.691	0.388 9	0.046 9
+0.54	+0.759 2	−0.140 47	−0.612 1	−0.004 3	2.201	2.061	0.547	3.738	0.652	0.376 0	0.045 7
+0.56	+0.802 6	−0.151 89	−0.655 1	+0.014 3	2.135	2.239	0.456	3.789	0.609	0.361 5	0.044 5
+0.58	+0.844 6	−0.164 13	−0.702 0	+0.022 5	2.063	2.431	0.372	3.831	0.561	0.345 5	0.043 0
+0.59	+0.865 0	−0.170 58	−0.726 9	+0.026 1	2.025	2.531	0.334	3.852	0.546	0.336 8	0.042 3
+0.60	+0.885 1	−0.177 27	−0.752 7	+0.029 2	1.986	2.635	0.297	3.868	0.509	0.327 7	0.041 6
+0.61	+0.901 7	−0.184 21	−0.779 6	+0.032 0	1.946	2.743	0.263	3.879	0.481	0.318 1	0.040 8
+0.62	+0.921 0	−0.191 43	−0.807 6	+0.034 5	1.905	2.854	0.232	3.885	0.452	0.308 2	0.040 0
+0.63	+0.942 8	−0.198 92	−0.836 7	+0.036 7	1.863	2.969	0.202	3.886	0.422	0.297 8	0.039 2
+0.64	+0.961 3	−0.206 72	−0.867 0	+0.038 6	1.820	3.087	0.175	3.878	0.391	0.287 0	0.038 4
+0.65	+0.979 2	−0.214 85	−0.898 1	+0.040 2	1.777	3.211	0.151	3.861	0.359	0.275 9	0.037 5
+0.66	+0.996 8	−0.223 36	−0.931 2	+0.041 6	1.733	3.338	0.129	3.832	0.326	0.264 1	0.036 6
+0.67	+1.013 9	−0.232 29	−0.965 2	+0.042 8	1.688	3.470	0.109	3.789	0.293	0.252 7	0.035 8
+0.68	+1.030 5	−0.241 71	−1.000 6	+0.043 8	1.643	3.609	0.091	3.728	0.259	0.240 8	0.034 9
+0.69	+1.046 7	−0.251 68	−1.037 1	+0.044 6	1.598	3.753	0.075	3.646	0.226	0.229 0	0.034 1
+0.70	+1.062 5	−0.262 33	−1.075 7	+0.045 3	1.552	3.904	0.061	3.538	0.193	0.217 3	0.033 3
+0.705	+1.070 2	−0.267 96	−1.095 1	+0.045 6	1.529	3.982	0.055	3.472	0.177	0.211 7	0.033 0
+0.710	+1.077 8	−0.273 82	−1.115 5	+0.045 8	1.506	4.064	0.049	3.396	0.162	0.206 2	0.032 6
+0.715	+1.085 2	−0.279 91	−1.136 1	+0.046 1	1.484	4.149	0.044	3.310	0.147	0.201 0	0.032 2
+0.720	+1.092 6	−0.286 37	−1.157 0	+0.046 3	1.462	4.237	0.039	3.213	0.133	0.196 1	0.031 9
+0.725	+1.099 9	−0.293 15	−1.178 1	+0.046 5	1.440	4.329	0.034	3.102	0.121	0.191 6	0.031 6

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.730	+1.107 0	−0.300 35	−1.200 3	+0.046 6	1.418	4.425	0.030	2.977	0.110	0.187 7	0.031 4
+0.735	+1.111 0	−0.308 05	−1.222 7	+0.046 8	1.397	4.528	0.026	2.835	0.101	0.184 6	0.031 2
+0.740	+1.121 0	−0.316 36	−1.245 6	+0.046 9	1.376	4.638	0.023	2.674	0.095	0.182 4	0.031 0
+0.742	+1.123 7	−0.319 90	−1.255 0	+0.046 9	1.369	4.684	0.021	2.603	0.094	0.182 0	0.031 0
+0.744	+1.126 1	−0.323 56	−1.261 1	+0.047 0	1.361	4.732	0.020	2.530	0.093	0.181 7	0.031 0
+0.746	+1.129 2	−0.327 38	−1.273 9	+0.047 0	1.353	4.781	0.019	2.451	0.093	0.181 7	0.031 0

1. 3

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.50	+0.667 9	−0.118 62	−0.536 3	−0.020 5	2.265	1.733	0.660	3.696	0.718	0.398 1	0.047 7
+0.52	+0.713 8	−0.128 37	−0.572 5	−0.006 1	2.235	1.886	0.605	3.761	0.682	0.386 1	0.046 6
+0.54	+0.758 5	−0.138 68	−0.611 8	+0.005 7	2.203	2.050	0.556	3.836	0.642	0.372 7	0.045 1
+0.56	+0.801 9	−0.149 71	−0.654 5	+0.015 9	2.134	2.226	0.466	3.910	0.597	0.357 5	0.044 1
+0.58	+0.843 8	−0.161 16	−0.700 9	+0.024 3	2.061	2.414	0.383	3.986	0.547	0.340 5	0.042 6
+0.60	+0.881 2	−0.173 91	−0.751 2	+0.031 2	1.981	2.611	0.308	4.063	0.491	0.321 5	0.041 0
+0.62	+0.923 2	−0.187 20	−0.805 5	+0.036 7	1.902	2.825	0.243	4.139	0.429	0.300 2	0.039 1
+0.64	+0.960 3	−0.201 25	−0.861 2	+0.041 0	1.816	3.048	0.186	4.211	0.361	0.276 7	0.037 6
+0.66	+0.995 8	−0.216 16	−0.927 5	+0.044 2	1.726	3.283	0.140	4.281	0.287	0.250 7	0.035 7
+0.68	+1.029 1	−0.231 95	−0.995 6	+0.046 6	1.634	3.529	0.102	4.340	0.206	0.222 0	0.033 7
+0.70	+1.061 1	−0.248 71	−1.068 8	+0.048 3	1.539	3.787	0.072	4.387	0.118	0.190 8	0.031 6
+0.72	+1.091 0	−0.266 51	−1.147 [illegible]	+0.049 5	1.442	4.055	0.049	4.444	0.025	0.156 9	0.029 5
+0.73	+1.105 1	−0.275 85	−1.188 1	+0.050 0	1.393	4.195	0.040	4.447	−0.025	0.139 0	0.028 1
+0.74	+1.118 8	−0.285 [illegible]1	−1.231 0	+0.050 3	1.341	4.335	0.032	4.442	−0.075	0.120 5	0.027 3
+0.75	+1.132 0	−0.295 52	−1.275 1	+0.050 6	1.295	4.480	0.026	4.395	−0.127	0.101 5	0.026 2
+0.76	+1.144 7	−0.305 91	−1.320 7	+0.050 8	1.246	4.628	0.021	4.367	−0.179	0.081 9	0.025 2
+0.77	+1.156 9	−0.316 71	−1.367 7	+0.051 0	1.196	4.780	0.016	4.321	−0.233	0.062 0	0.024 2
+0.78	+1.168 6	−0.328 06	−1.416 3	+0.051 2	1.148	4.937	0.013	4.256	−0.287	0.041 7	0.023 2
+0.79	+1.179 9	−0.339 98	−1.466 5	+0.051 3	1.100	5.101	0.010	4.161	−0.341	0.021 3	0.022 2
+0.80	+1.190 6	−0.352 61	−1.518 3	+0.051 4	1.053	5.273	0.007	4.010	−0.391	0.001 0	0.021 2
+0.805	+1.195 8	−0.359 25	−1.541 9	+0.051 4	1.030	5.361	0.006	3.964	−0.419	−0.009 0	0.020 8
+0.810	+1.200 9	−0.366 15	−1.571 9	+0.051 4	1.007	5.453	0.005	3.876	−0.445	−0.018 8	0.020 1
+0.815	+1.205 9	−0.373 51	−1.599 5	+0.051 5	0.984	5.549	0.005	3.775	−0.470	−0.028 3	0.020 0
+0.820	+1.210 8	−0.380 87	−1.627 5	+0.051 5	0.962	5.650	0.004	3.659	−0.494	−0.037 5	0.019 6
+0.825	+1.215 5	−0.388 80	−1.656 0	+0.051 5	0.940	5.756	0.003	3.525	−0.516	−0.046 2	0.019 2
+0.830	+1.220 [illegible]	−0.397 [illegible]	−1.685 0	+0.051 5	0.919	5.868	0.003	3.371	−0.537	−0.054 2	0.018 9
+0.835	+1.224 [illegible]	−0.406 [illegible]	−1.714 7	+0.051 5	0.898	5.989	0.002	3.195	−0.555	−0.061 3	0.018 6
+0.840	+1.229 1	−0.416 03	−1.745 9	+0.051 6	0.878	6.120	0.002	2.992	−0.570	−0.067 1	0.018 1

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.842	+1.230 9	−0.420 21	−1.757 2	+0.051 6	0.871	6.178	0.002	2.902	−0.575	−0.068 9	0.018 3
+0.844	+1.232 6	−0.424 56	−1.769 6	+0.051 6	0.863	6.237	0.002	2.807	−0.579	−0.070 5	0.018 2
+0.846	+1.231 3	−0.429 11	−1.782 2	+0.051 6	0.856	6.299	0.001	2.706	−0.582	−0.071 6	0.018 2
+0.848	+1.236 0	−0.433 88	−1.791 8	+0.051 6	0.849	6.364	0.001	2.599	−0.584	−0.072 1	0.018 2
+0.850	+1.237 7	−0.438 91	−1.807 6	+0.051 6	0.842	6.431	0.001	2.486	−0.581	−0.072 6	0.018 2

[1. 4]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.54	+0.758 4	−0.138 50	−0.611 7	+0.005 8	2.203	2.049	0.557	3.846	0.641	0.372 4	0.045 4
+0.56	+0.801 8	−0.149 49	−0.651 5	+0.016 0	2.134	2.224	0.467	3.923	0.596	0.357 1	0.044 1
+0.58	+0.843 8	−0.161 19	−0.700 8	+0.024 5	2.061	2.412	0.381	4.002	0.515	0.340 0	0.042 6
+0.60	+0.884 2	−0.173 61	−0.751 0	+0.031 4	1.983	2.611	0.309	4.083	0.489	0.320 9	0.041 0
+0.62	+0.923 1	−0.186 78	−0.805 3	+0.036 9	1.902	2.822	0.244	4.164	0.427	0.299 5	0.039 3
+0.64	+0.960 3	−0.200 72	−0.861 0	+0.041 2	1.816	3.044	0.188	4.245	0.359	0.275 7	0.037 5
+0.66	+0.995 7	−0.215 17	−0.927 1	+0.044 5	1.726	3.277	0.141	4.324	0.283	0.249 4	0.035 6
+0.68	+1.029 3	−0.231 05	−0.995 1	+0.046 9	1.634	3.522	0.103	4.400	0.201	0.220 4	0.033 5
+0.70	+1.061 0	−0.247 51	−1.068 1	+0.048 6	1.538	3.777	0.073	4.468	0.112	0.188 5	0.031 4
+0.72	+1.090 8	−0.264 88	−1.146 2	+0.049 9	1.441	4.040	0.050	4.528	0.017	0.153 8	0.029 3
+0.74	+1.118 6	−0.283 22	−1.229 7	+0.050 7	1.342	4.313	0.033	4.575	−0.087	0.116 2	0.027 1
+0.75	+1.131 8	−0.292 78	−1.273 6	+0.051 0	1.292	4.452	0.027	4.591	−0.141	0.096 2	0.026 0
+0.76	+1.144 5	−0.302 62	−1.318 8	+0.051 3	1.243	4.594	0.021	4.601	−0.196	0.075 6	0.024 9
+0.77	+1.156 6	−0.312 75	−1.365 5	+0.051 5	1.191	4.738	0.017	4.605	−0.253	0.054 3	0.023 8
+0.78	+1.168 3	−0.323 21	−1.413 6	+0.051 6	1.145	4.885	0.013	4.600	−0.311	0.032 3	0.022 7
+0.79	+1.179 5	−0.334 00	−1.463 2	+0.051 7	1.096	5.034	0.010	4.585	−0.371	0.009 7	0.021 6
+0.80	+1.190 3	−0.345 18	−1.514 3	+0.051 8	1.047	5.186	0.008	4.557	−0.400	−0.013 5	0.020 6
+0.81	+1.200 5	−0.356 78	−1.566 9	+0.051 9	1.000	5.341	0.006	4.514	−0.493	−0.037 1	0.019 6
+0.82	+1.210 3	−0.368 88	−1.621 1	+0.051 9	0.953	5.502	0.005	4.449	−0.555	−0.061 1	0.018 6
+0.83	+1.219 6	−0.381 55	−1.677 0	+0.052 0	0.906	5.668	0.003	4.360	−0.617	−0.085 2	0.017 7
+0.84	+1.228 4	−0.391 93	−1.731 5	+0.052 0	0.862	5.841	0.002	4.238	−0.679	−0.109 3	0.016 8
+0.85	+1.236 8	−0.409 20	−1.793 8	+0.052 0	0.818	6.027	0.002	4.072	−0.739	−0.133 0	0.015 9
+0.86	+1.244 8	−0.424 64	−1.855 1	+0.052 0	0.776	6.226	0.001	3.853	−0.796	−0.155 5	0.015 2
+0.865	+1.248 6	−0.432 93	−1.886 5	+0.052 1	0.755	6.335	0.001	3.717	−0.825	−0.166 3	0.014 8
+0.870	+1.252 3	−0.441 72	−1.918 5	+0.052 1	0.735	6.450	0.001	3.560	−0.851	−0.176 3	0.014 5
+0.875	+1.256 0	−0.451 10	−1.951 0	+0.052 1	0.716	6.571	0.001	3.379	−0.871	−0.185 1	0.014 2
+0.880	+1.259 5	−0.461 23	−1.981 2	+0.052 1	0.697	6.710	0.001	3.169	−0.893	−0.193 3	0.014 0
+0.885	+1.262 9	−0.472 37	−2.018 1	+0.052 1	0.680	6.863	0.000	2.921	−0.909	−0.199 4	0.013 8
+0.890	+1.266 3	−0.481 85	−2.052 9	+0.052 1	0.661	7.038	0.000	2.610	−0.918	−0.202 9	0.013 7
+0.892 5	+1.267 9	−0.491 78	−2.070 6	+0.052 1	0.656	7.137	0.000	2.179	−0.919	−0.203 3	0.013 7

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
					[1. 5]						
+0.83	+1.219 3	−0.375 55	−1.673 5	+0.052 2	0.903	5.592	0.004	4.803	−0.646	−0.096 7	0.017 3
+0.84	+1.228 1	−0.387 37	−1.730 1	+0.052 2	0.857	5.711	0.003	4.789	−0.716	−0.123 8	0.016 2
+0.85	+1.236 4	−0.399 56	−1.788 1	+0.052 2	0.812	5.898	0.002	4.763	−0.786	−0.151 6	0.015 3
+0.86	+1.244 3	−0.412 15	−1.848 1	+0.052 3	0.768	6.055	0.001	4.723	−0.860	−0.179 9	0.014 4
+0.87	+1.251 8	−0.425 22	−1.909 5	+0.052 3	0.725	6.216	0.001	4.661	−0.932	−0.208 5	0.013 5
+0.88	+1.258 8	−0.438 81	−1.972 5	+0.052 3	0.684	6.382	0.001	4.580	−1.002	−0.237 5	0.012 7
+0.89	+1.265 5	−0.453 14	−2.037 1	+0.052 3	0.644	6.557	0.001	4.464	−1.071	−0.266 1	0.011 9
+0.90	+1.271 7	−0.468 29	−2.103 6	+0.052 3	0.605	6.712	0.000	4.307	−1.145	−0.295 0	0.011 2
+0.905	+1.274 7	−0.476 26	−2.137 6	+0.052 3	0.586	6.842	0.000	4.207	−1.181	−0.309 0	0.010 8
+0.910	+1.277 6	−0.484 57	−2.172 1	+0.052 3	0.568	6.945	0.000	4.091	−1.214	−0.322 8	0.010 5
+0.915	+1.280 1	−0.493 25	−2.207 0	+0.052 3	0.550	7.051	0.000	3.960	−1.246	−0.335 8	0.010 2
+0.920	+1.283 1	−0.502 41	−2.242 6	+0.052 3	0.533	7.171	0.000	3.805	−1.277	−0.348 7	0.009 9
+0.925	+1.285 7	−0.512 11	−2.278 8	+0.052 3	0.515	7.297	0.000	3.625	−1.306	−0.360 5	0.009 7
+0.930	+1.288 3	−0.522 60	−2.315 6	+0.052 3	0.500	7.437	0.000	3.415	−1.332	−0.371 2	0.009 5
+0.935	+1.290 7	−0.534 01	−2.353 2	+0.052 3	0.485	7.593	0.000	3.168	−1.355	−0.380 3	0.009 3
+0.940	+1.293 1	−0.546 71	−2.391 6	+0.052 3	0.471	7.775	0.000	2.878	−1.371	−0.386 9	0.009 1
+0.942	+1.294 1	−0.552 27	−2.407 2	+0.052 3	0.466	7.857	0.000	2.718	−1.375	−0.388 7	0.009 1
+0.944	+1.295 0	−0.558 17	−2.423 0	+0.052 3	0.461	7.944	0.000	2.609	−1.378	−0.389 7	0.009 1
+0.946	+1.295 9	−0.564 17	−2.439 0	+0.052 3	0.456	8.011	0.000	2.460	−1.378	−0.389 9	0.009 1
					[1. 6]						
+0.88	+1.258 7	−0.435 71	−1.970 3	+0.052 3	0.682	6.330	0.001	4.858	−1.019	−0.244 1	0.012 5
+0.89	+1.265 3	−0.448 [illegible]	−2.031 1	+0.052 3	0.641	6.488	0.001	4.817	−1.096	−0.275 3	0.011 7
+0.90	+1.271 6	−0.462 16	−2.100 1	+0.052 3	0.602	6.650	0.000	4.760	−1.171	−0.306 8	0.010 9
+0.91	+1.277 4	−0.476 13	−2.167 1	+0.052 3	0.561	6.817	0.000	4.675	−1.252	−0.338 5	0.010 2
+0.92	+1.282 9	−0.491 37	−2.236 1	+0.052 3	0.528	6.994	0.000	4.560	−1.330	−0.370 1	0.009 5
+0.93	+1.288 0	−0.507 18	−2.306 3	+0.052 3	0.493	7.184	0.000	4.401	−1.406	−0.401 5	0.008 8
+0.94	+1.292 7	−0.524 13	−2.380 2	+0.052 3	0.463	7.390	0.000	4.185	−1.481	−0.432 0	0.008 2
+0.945	+1.295 0	−0.533 17	−2.417 1	+0.052 3	0.444	7.502	0.000	4.050	−1.516	−0.446 6	0.008 0
+0.950	+1.297 2	−0.542 69	−2.455 2	+0.052 3	0.429	7.622	0.000	3.891	−1.550	−0.460 6	0.007 8
+0.955	+1.299 3	−0.552 81	−2.493 6	+0.052 3	0.414	7.755	0.000	3.701	−1.582	−0.473 7	0.007 5
+0.960	+1.301 3	−0.563 69	−2.532 8	+0.052 3	0.400	7.901	0.000	3.486	−1.611	−0.485 7	0.007 3

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.962	+1.302 1	−0.568 30	−2.518 6	+0.052 3	0.395	7.964	0.000	3.388	−1.621	−0.490 0	0.007 3
+0.964	+1.302 9	−0.573 10	−2.561 6	+0.052 3	0.389	8.032	0.000	3.283	−1.631	−0.494 0	0.007 2
+0.966	+1.303 7	−0.578 10	−2.580 8	+0.052 3	0.384	8.104	0.000	3.172	−1.639	−0.497 7	0.007 1
+0.968	+1.304 1	−0.583 33	−2.597 1	+0.052 3	0.379	8.179	0.000	3.052	−1.647	−0.500 8	0.007 1
+0.970	+1.305 2	−0.588 83	−2.613 5	+0.052 3	0.374	8.259	0.000	2.924	−1.654	−0.503 5	0.007 1
+0.972	+1.306 0	−0.594 64	−2.630 1	+0.052 3	0.370	8.349	0.000	2.787	−1.659	−0.505 6	0.007 0
+0.974	+1.306 7	−0.600 83	−2.646 9	+0.052 3	0.365	8.434	0.000	2.610	−1.662	−0.507 0	0.007 0
+0.976	+1.307 1	−0.607 17	−2.663 9	+0.052 3	0.361	8.519	0.000	2.481	−1.663	−0.507 4	0.007 0

[1. 7]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.96	+1.300 9	−0.539 15	−2.518 8	+0.052 4	0.392	7.473	0.000	5.211	−1.721	−0.532 9	0.006 6
+0.97	+1.304 7	−0.553 62	−2.594 3	+0.052 4	0.362	7.626	0.000	5.222	−1.818	−0.572 2	0.006 0
+0.98	+1.308 2	−0.568 36	−2.671 3	+0.052 4	0.334	7.781	0.000	5.232	−1.915	−0.612 6	0.005 5
+0.99	+1.311 1	−0.583 36	−2.749 9	+0.052 4	0.307	7.939	0.000	5.242	−2.013	−0.653 6	0.005 0
+1.00	+1.314 3	−0.598 65	−2.830 1	+0.052 4	0.282	8.098	0.000	5.251	−2.113	−0.695 6	0.004 6
+1.01	+1.317 0	−0.614 21	−2.911 9	+0.052 4	0.258	8.263	0.000	5.259	−2.214	−0.738 5	0.004 1
+1.02	+1.319 5	−0.630 07	−2.995 4	+0.052 4	0.237	8.431	0.000	5.266	−2.318	−0.782 3	0.003 7
+1.03	+1.321 7	−0.646 23	−3.080 6	+0.052 4	0.214	8.601	0.000	5.273	−2.423	−0.827 1	0.003 4
+1.04	+1.323 8	−0.662 70	−3.167 1	+0.052 4	0.194	8.777	0.000	5.278	−2.531	−0.872 8	0.003 1
+1.05	+1.325 6	−0.679 50	−3.256 1	+0.052 4	0.176	8.960	0.000	5.282	−2.640	−0.919 5	0.002 8
+1.06	+1.327 3	−0.696 63	−3.346 6	+0.052 4	0.159	9.150	0.000	5.286	−2.752	−0.967 2	0.002 5
+1.07	+1.328 8	−0.714 12	−3.439 1	+0.052 4	0.143	9.348	0.000	5.287	−2.866	−1.015 9	0.002 2
+1.08	+1.330 2	−0.732 00	−3.533 6	+0.052 4	0.128	9.556	0.000	5.287	−2.982	−1.065 7	0.001 9
+1.09	+1.331 4	−0.750 28	−3.630 3	+0.052 4	0.114	9.771	0.000	5.285	−3.101	−1.116 7	0.001 7
+1.10	+1.332 5	−0.768 99	−3.729 1	+0.052 4	0.101	9.996	0.000	5.280	−3.222	−1.168 7	0.001 6
+1.11	+1.333 4	−0.788 16	−3.830 2	+0.052 4	0.090	10.237	0.000	5.270	−3.346	−1.221 9	0.001 4
+1.12	+1.334 3	−0.807 85	−3.933 9	+0.052 4	0.079	10.493	0.000	5.251	−3.472	−1.276 3	0.001 2
+1.13	+1.335 0	−0.828 13	−4.040 2	+0.052 4	0.070	10.765	0.000	5.229	−3.601	−1.332 0	0.001 1
+1.135	+1.335 3	−0.838 51	−4.094 1	+0.052 4	0.065	10.910	0.000	5.211	−3.662	−1.358 1	0.001 0
+1.140	+1.335 6	−0.849 07	−4.149 3	+0.052 4	0.061	11.060	0.000	5.190	−3.733	−1.388 7	0.001 0
+1.145	+1.335 9	−0.859 82	−4.205 0	+0.052 4	0.057	11.216	0.000	5.163	−3.799	−1.417 5	0.000 9
+1.150	+1.336 2	−0.870 80	−4.261 5	+0.052 4	0.053	11.381	0.000	5.129	−3.867	−1.446 6	0.000 8
+1.155	+1.336 5	−0.882 02	−4.318 8	+0.052 4	0.049	11.553	0.000	5.087	−3.934	−1.475 8	0.000 8
+1.160	+1.336 7	−0.893 52	−4.377 0	+0.052 4	0.046	11.735	0.000	5.036	−4.003	−1.505 3	0.000 7
+1.165	+1.336 9	−0.905 31	−4.436 2	+0.052 4	0.042	11.928	0.000	4.970	−4.071	−1.534 9	0.000 7
+1.170	+1.337 1	−0.917 51	−4.496 3	+0.052 4	0.039	12.135	0.000	4.889	−4.140	−1.561 5	0.000 6
+1.175	+1.337 3	−0.930 19	−4.557 5	+0.052 4	0.036	12.357	0.000	4.786	−4.208	−1.591 2	0.000 6
+1.180	+1.337 5	−0.943 40	−4.619 9	+0.052 4	0.034	12.598	0.000	4.656	−4.276	−1.623 5	0.000 6

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+1.182	+1.337 5	−0.948 87	−4.645 2	+0.052 4	0.032	12.703	0.000	4.595	−4.303	−1.635 1	0.000 5
+1.184	+1.337 6	−0.954 46	−4.670 7	+0.052 4	0.031	12.810	0.000	4.527	−4.329	−1.646 7	0.000 5
+1.186	+1.337 7	−0.960 19	−4.696 4	+0.052 4	0.031	12.923	0.000	4.453	−4.356	−1.658 1	0.000 5
+1.188	+1.337 7	−0.966 08	−4.722 4	+0.052 4	0.029	13.041	0.000	4.371	−4.382	−1.669 3	0.000 5
+1.190	+1.337 8	−0.972 14	−4.748 6	+0.052 4	0.029	13.166	0.000	4.280	−4.407	−1.680 4	0.000 5
+1.192	+1.337 8	−0.978 39	−4.775 1	+0.052 4	0.028	13.298	0.000	4.180	−4.432	−1.691 2	0.000 5
+1.194	+1.337 9	−0.984 87	−4.801 8	+0.052 4	0.027	13.438	0.000	4.069	−4.457	−1.701 7	0.000 5
+1.196	+1.338 0	−0.991 61	−4.828 8	+0.052 4	0.026	13.587	0.000	3.946	−4.480	−1.711 9	0.000 4
+1.198	+1.338 0	−0.998 66	−4.856 2	+0.052 4	0.025	13.748	0.000	3.809	−4.503	−1.721 6	0.000 4
+1.200	+1.338 1	−1.006 07	−4.883 8	+0.052 4	0.024	13.922	0.000	3.657	−4.524	−1.730 7	0.000 4
+1.202	+1.338 1	−1.013 92	−4.911 9	+0.052 4	0.023	14.113	0.000	3.487	−4.543	−1.739 2	0.000 4
+1.204	+1.338 2	−1.022 30	−4.940 3	+0.052 4	0.023	14.324	0.000	3.297	−4.560	−1.746 7	0.000 4
+1.206	+1.338 2	−1.031 35	−4.969 2	+0.052 4	0.022	14.560	0.000	3.083	−4.575	−1.752 9	0.000 4
+1.207	+1.338 2	−1.036 18	−4.983 8	+0.052 4	0.022	14.689	0.000	2.967	−4.581	−1.755 5	0.000 4
+1.208	+1.338 2	−1.041 26	−4.998 6	+0.052 4	0.021	14.829	0.000	2.843	−4.586	−1.757 5	0.000 4
+1.209	+1.338 3	−1.046 63	−5.013 5	+0.052 4	0.021	14.980	0.000	2.712	−4.589	−1.759 0	0.000 4
+1.210	+1.338 3	−1.052 33	−5.028 5	+0.052 4	0.021	15.143	0.000	2.572	−4.591	−1.759 8	0.000 4
+1.211	+1.338 3	−1.058 43	−5.043 8	+0.052 4	0.020	15.323	0.000	2.423	−4.591	−1.759 8	0.000 4

[1.8]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.85	+1.236 3	−0.395 91	−1.786 2	+0.052 3	0.810	5.849	0.002	5.053	−0.806	−0.158 5	0.015 1
+0.86	+1.244 [illegible]	−0.407 59	−1.845 1	+0.052 3	0.765	5.993	0.002	5.085	−0.881	−0.188 5	0.014 1
+0.87	+1.251 6	−0.419 48	−1.906 0	+0.052 3	0.722	6.136	0.001	5.118	−0.959	−0.219 5	0.013 2
+0.88	+1.258 6	−0.431 57	−1.968 1	+0.052 4	0.680	6.279	0.001	5.153	−1.037	−0.251 3	0.012 3
+0.99	+1.265 2	−0.443 84	−2.031 6	+0.052 4	0.639	6.420	0.001	5.191	−1.118	−0.284 1	0.011 5
+0.90	+1.271 4	−0.456 30	−2.096 5	+0.052 4	0.600	6.563	0.000	5.232	−1.202	−0.317 9	0.010 6
+0.91	+1.277 2	−0.468 92	−2.162 9	+0.052 4	0.561	6.704	0.000	5.280	−1.287	−0.352 6	0.009 8
+0.92	+1.282 6	−0.481 68	−2.230 6	+0.052 4	0.524	6.842	0.000	5.336	−1.375	−0.388 5	0.009 1
+0.93	+1.287 7	−0.494 56	−2.299 7	+0.052 4	0.488	6.980	0.000	5.402	−1.465	−0.425 4	0.008 4
+0.94	+1.292 4	−0.507 51	−2.370 2	+0.052 4	0.454	7.115	0.000	5.483	−1.557	−0.463 5	0.007 7

[1. 9]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.70	+1.060 9	−0.246 35	−1.067 5	+0.049 0	1.537	3.766	0.074	4.549	0.106	0.186 3	0.031 3
+0.72	+1.090 6	−0.263 30	−1.145 4	+0.050 2	1.440	4.026	0.051	4.641	0.008	0.150 8	0.029 1
+0.74	+1.118 1	−0.281 01	−1.228 5	+0.051 0	1.341	4.292	0.031	4.735	−0.098	0.112 0	0.026 8
+0.76	+1.144 2	−0.299 55	−1.317 1	+0.051 6	1.241	4.562	0.022	4.832	−0.112	0.069 7	0.024 6

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.77	+1.156 4	−0.309 08	−1.363 4	+0.051 8	1.191	4.699	0.018	4.881	−0.272	0.017 3	0.023 5
+0.78	+1.168 1	−0.318 79	−1.411 0	+0.051 9	1.141	4.837	0.014	4.938	−0.334	0.023 9	0.022 3
+0.79	+1.179 2	−0.328 67	−1.460 1	+0.052 1	1.092	4.975	0.011	4.995	−0.395	−0.000 5	0.021 1
+0.80	+1.189 9	−0.338 70	−1.510 5	+0.052 2	1.043	5.111	0.009	5.058	−0.463	−0.025 9	0.020 1
+0.81	+1.200 1	−0.348 87	−1.562 3	+0.052 2	0.994	5.248	0.007	5.127	−0.532	−0.052 2	0.019 0
+0.82	+1.209 8	−0.359 17	−1.615 5	+0.052 3	0.946	5.383	0.005	5.204	−0.603	−0.079 6	0.017 9
+0.83	+1.219 0	−0.369 56	−1.670 0	+0.052 3	0.899	5.518	0.004	5.291	−0.676	−0.108 2	0.016 8
+0.84	+1.227 8	−0.380 01	−1.725 8	+0.052 4	0.852	5.649	0.003	5.394	−0.751	−0.137 9	0.015 8
					[1.10]						
+0.32	+0.210 9	−0.055 95	−0.320 5	−0.243 2	2.698	0.782 3	1.704	3.266	0.901	0.457 8	0.053 5
+0.34	+0.264 6	−0.060 93	−0.336 9	−0.210 1	2.669	0.856 0	1.609	3.304	0.889	0.453 9	0.053 1
+0.36	+0.317 7	−0.066 32	−0.354 8	−0.178 9	2.638	0.936 3	1.510	3.346	0.876	0.449 4	0.052 6
+0.38	+0.370 1	−0.072 13	−0.374 4	−0.149 7	2.604	1.024	1.407	3.391	0.860	0.444 4	0.052 1
+0.40	+0.421 8	−0.078 40	−0.395 8	−0.122 6	2.566	1.119	1.301	3.441	0.842	0.438 7	0.051 6
+0.42	+0.472 7	−0.083 14	−0.419 2	−0.097 7	2.526	1.222	1.194	3.495	0.822	0.432 1	0.050 9
+0.44	+0.522 8	−0.092 39	−0.444 7	−0.074 9	2.482	1.334	1.085	3.554	0.799	0.424 7	0.050 2
+0.46	+0.572 0	−0.100 16	−0.472 5	−0.054 3	2.431	1.455	0.977	3.618	0.771	0.416 3	0.049 4
+0.48	+0.620 1	−0.108 47	−0.502 9	−0.035 8	2.382	1.585	0.869	3.688	0.745	0.406 7	0.048 5
+0.50	+0.667 2	−0.117 36	−0.536 0	−0.019 5	2.326	1.726	0.764	3.764	0.711	0.397 8	0.047 5
+0.52	+0.713 1	−0.126 82	−0.572 0	−0.005 2	2.266	1.877	0.662	3.846	0.674	0.383 4	0.046 1
+0.54	+0.757 8	−0.136 89	−0.611 2	+0.007 0	2.202	2.039	0.566	3.936	0.625	0.369 5	0.045 2
+0.56	+0.801 2	−0.147 55	−0.653 7	+0.017 5	2.133	2.213	0.476	4.034	0.585	0.353 6	0.043 8
					[1.11]						
−0.62	−2.545 0	−0.000 79	−0.152 6	−2.755 8	2.995	0.010 45	2.979	2.896	0.999	0.489 0	0.056 8
−0.60	−2.485 1	−0.000 86	−0.152 8	−2.696 3	2.995	0.011 45	2.976	2.896	0.999	0.489 0	0.056 8
−0.58	−2.425 2	−0.000 94	−0.153 0	−2.636 8	2.994	0.012 56	2.974	2.897	0.999	0.489 0	0.056 8
−0.56	−2.365 3	−0.001 03	−0.153 3	−2.577 3	2.994	0.013 77	2.972	2.898	0.999	0.488 9	0.056 8
−0.54	−2.305 4	−0.001 13	−0.153 6	−2.517 9	2.994	0.015 10	2.969	2.899	0.999	0.488 9	0.056 8
−0.52	−2.245 5	−0.001 24	−0.153 9	−2.458 6	2.993	0.016 55	2.967	2.899	0.999	0.488 8	0.056 8
−0.50	−2.185 7	−0.001 36	−0.154 2	−2.399 3	2.992	0.018 15	2.963	2.900	0.998	0.488 8	0.056 8
−0.48	−2.125 8	−0.001 49	−0.154 6	−2.340 0	2.992	0.019 90	2.960	2.901	0.998	0.488 7	0.056 8
−0.46	−2.066 0	−0.001 61	−0.155 0	−2.280 9	2.991	0.021 82	2.956	2.902	0.998	0.488 7	0.056 8
−0.44	−2.006 2	−0.001 79	−0.155 5	−2.221 8	2.990	0.023 92	2.952	2.903	0.998	0.488 6	0.056 8
−0.42	−1.946 4	−0.001 97	−0.156 0	−2.162 8	2.990	0.026 23	2.947	2.904	0.998	0.488 5	0.056 8
−0.40	−1.886 6	−0.002 16	−0.156 5	−2.103 9	2.989	0.028 76	2.943	2.905	0.997	0.488 5	0.056 7

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
−0.38	−1.826 8	−0.002 36	−0.157 1	−2.015 1	2.988	0.031 54	2.938	2.907	0.997	0.488 4	0.056 7
−0.36	−1.767 1	−0.002 59	−0.157 8	−1.986 4	2.987	0.034 58	2.931	2.908	0.997	0.488 3	0.056 7
−0.34	−1.707 4	−0.002 84	−0.158 5	−1.927 9	2.985	0.037 91	2.925	2.910	0.996	0.488 2	0.056 7
−0.32	−1.647 7	−0.003 11	−0.159 3	−1.869 1	2.984	0.041 57	2.917	2.912	0.996	0.488 1	0.056 7
−0.30	−1.588 0	−0.003 41	−0.160 2	−1.811 2	2.982	0.045 57	2.909	2.914	0.996	0.487 9	0.056 7
−0.28	−1.528 4	−0.003 74	−0.161 1	−1.753 1	2.981	0.049 96	2.900	2.916	0.995	0.487 8	0.056 7
−0.26	−1.468 8	−0.004 10	−0.162 2	−1.695 2	2.979	0.054 78	2.891	2.919	0.995	0.487 7	0.056 7
−0.24	−1.409 3	−0.004 49	−0.163 3	−1.637 5	2.977	0.060 07	2.882	2.921	0.994	0.487 5	0.056 6
−0.22	−1.349 7	−0.004 92	−0.164 6	−1.580 0	2.975	0.065 86	2.870	2.924	0.994	0.487 3	0.056 6
−0.20	−1.290 3	−0.005 39	−0.166 0	−1.522 7	2.972	0.072 20	2.857	2.928	0.993	0.487 1	0.056 6
−0.18	−1.230 9	−0.005 91	−0.167 5	−1.465 7	2.969	0.079 15	2.844	2.931	0.992	0.486 9	0.056 6
−0.16	−1.171 5	−0.006 47	−0.169 2	−1.409 0	2.966	0.086 78	2.829	2.935	0.992	0.486 6	0.056 6
−0.14	−1.112 2	−0.007 09	−0.171 0	−1.352 5	2.964	0.095 15	2.813	2.940	0.991	0.486 4	0.056 5
−0.12	−1.053 0	−0.007 77	−0.173 0	−1.296 5	2.960	0.104 3	2.796	2.945	0.990	0.486 1	0.056 5
−0.10	−0.993 8	−0.008 51	−0.175 2	−1.240 7	2.956	0.114 4	2.776	2.950	0.989	0.485 7	0.056 5
−0.08	−0.934 7	−0.009 32	−0.177 6	−1.185 4	2.951	0.125 9	2.755	2.956	0.988	0.485 4	0.056 4
−0.06	−0.875 7	−0.010 21	−0.180 2	−1.130 6	2.947	0.137 4	2.733	2.962	0.986	0.484 9	0.056 4
−0.04	−0.816 9	−0.011 18	−0.183 1	−1.076 2	2.942	0.150 7	2.708	2.969	0.985	0.484 5	0.056 3
−0.02	−0.758 1	−0.012 24	−0.186 2	−1.022 3	2.936	0.165 2	2.681	2.977	0.983	0.484 0	0.056 3
−0.00	−0.699 4	−0.013 40	−0.189 7	−0.968 9	2.930	0.181 1	2.652	2.985	0.982	0.483 4	0.056 2
+0.02	−0.640 9	−0.014 67	−0.193 5	−0.916 2	2.923	0.198 4	2.619	2.994	0.980	0.482 8	0.056 2
+0.04	−0.582 5	−0.016 05	−0.197 6	−0.864 2	2.916	0.217 5	2.585	3.004	0.977	0.482 1	0.056 1
+0.06	−0.524 2	−0.017 57	−0.202 2	−0.812 8	2.908	0.238 4	2.548	3.015	0.975	0.481 3	0.056 0
+0.08	−0.466 2	−0.019 22	−0.207 2	−0.762 3	2.899	0.261 3	2.507	3.027	0.972	0.480 5	0.055 9
+0.10	−0.408 3	−0.021 02	−0.212 7	−0.712 5	2.890	0.286 4	2.463	3.041	0.969	0.479 5	0.055 8
+0.12	−0.350 6	−0.022 99	−0.218 7	−0.663 7	2.879	0.313 9	2.416	3.055	0.966	0.478 4	0.055 7
+0.14	−0.293 1	−0.025 14	−0.225 2	−0.615 9	2.867	0.343 9	2.364	3.072	0.962	0.477 2	0.055 6
+0.16	−0.235 9	−0.027 48	−0.232 1	−0.569 2	2.855	0.376 9	2.310	3.089	0.958	0.475 9	0.055 4
+0.18	−0.179 0	−0.030 02	−0.240 3	−0.523 5	2.841	0.412 9	2.251	3.108	0.953	0.474 3	0.055 2
+0.20	−0.122 3	−0.032 79	−0.249 0	−0.479 2	2.825	0.452 3	2.187	3.130	0.948	0.472 6	0.055 1
+0.22	−0.066 0	−0.035 81	−0.258 4	−0.436 1	2.808	0.495 4	2.119	3.153	0.942	0.470 7	0.054 9
+0.24	−0.010 0	−0.039 08	−0.268 8	−0.394 1	2.790	0.542 5	2.047	3.179	0.935	0.468 5	0.054 7
+0.26	+0.046 0	−0.042 64	−0.280 2	−0.354 2	2.769	0.594 0	1.970	3.208	0.927	0.466 1	0.054 5
+0.28	+0.108 0	−0.046 50	−0.292 6	−0.315 6	2.748	0.650 3	1.888	3.239	0.918	0.463 3	0.054 2
+0.30	+0.155 5	−0.050 68	−0.306 2	−0.278 7	2.723	0.712 0	1.802	3.274	0.909	0.460 1	0.053 9
+0.32	+0.209 7	−0.055 20	−0.321 1	−0.243 6	2.696	0.780 2	1.711	3.320	0.897	0.456 5	0.053 5
+0.34	+0.263 3	−0.060 09	−0.337 4	−0.210 3	2.668	0.851 7	1.618	3.354	0.885	0.452 1	0.053 1
+0.36	+0.316 4	−0.065 36	−0.355 2	−0.178 9	2.636	0.931 3	1.520	3.401	0.871	0.447 8	0.052 7
+0.38	+0.368 8	−0.071 05	−0.374 7	−0.149 5	2.602	1.018	1.418	3.452	0.854	0.442 5	0.052 2
+0.40	+0.420 5	−0.077 16	−0.396 0	−0.122 2	2.565	1.112	1.314	3.509	0.836	0.436 5	0.051 6

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.42	+0.471 4	−0.083 73	−0.419 2	−0.097 0	2.525	1.214	1.208	3.571	0.815	0.429 7	0.050 8
+0.44	+0.521 5	−0.090 77	−0.444 5	−0.073 9	2.482	1.325	1.102	3.639	0.791	0.422 0	0.049 9
+0.46	+0.570 7	−0.098 30	−0.472 2	−0.053 0	2.434	1.445	0.994	3.716	0.764	0.413 1	0.049 1
+0.48	+0.618 8	−0.106 32	−0.502 4	−0.034 2	2.382	1.574	0.887	3.803	0.732	0.403 0	0.048 2
+0.50	+0.665 9	−0.114 85	−0.535 2	−0.017 5	2.325	1.712	0.782	3.900	0.699	0.391 5	0.047 2
+0.52	+0.711 8	−0.123 88	−0.570 9	−0.002 9	2.265	1.861	0.682	4.009	0.659	0.378 4	0.046 0
+0.54	+0.756 5	−0.133 41	−0.609 7	+0.009 8	2.201	2.020	0.586	4.133	0.615	0.363 1	0.044 6

[2. 1]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
−0.46	−2.049 5	−0.001 87	−0.166 8	−2.248 6	2.990	0.024 19	2.950	2.810	−0.002	0.147 2	0.028 9
−0.44	−1.989 7	−0.002 05	−0.167 3	−2.189 6	2.989	0.026 52	2.945	2.811	−0.002	0.147 1	0.028 9
−0.42	−1.929 9	−0.002 25	−0.167 8	−2.130 8	2.989	0.029 08	2.940	2.812	−0.002	0.147 0	0.028 9
−0.40	−1.870 2	−0.002 47	−0.168 5	−2.072 1	2.988	0.031 88	2.934	2.813	−0.002	0.147 0	0.028 9
−0.38	−1.810 4	−0.002 71	−0.169 1	−2.013 4	2.987	0.034 95	2.929	2.815	−0.002	0.146 9	0.028 9
−0.36	−1.750 7	−0.002 97	−0.169 9	−1.954 9	2.985	0.038 29	2.922	2.816	−0.003	0.146 8	0.028 9
−0.34	−1.691 0	−0.003 25	−0.170 7	−1.896 6	2.984	0.042 02	2.915	2.818	−0.003	0.146 7	0.028 8
−0.32	−1.631 4	−0.003 56	−0.171 5	−1.838 4	2.982	0.046 07	2.906	2.820	−0.003	0.146 6	0.028 8
−0.30	−1.571 7	−0.003 91	−0.172 5	−1.780 3	2.981	0.050 51	2.898	2.822	−0.003	0.146 5	0.028 8
−0.28	−1.512 2	−0.004 28	−0.173 6	−1.722 5	2.979	0.055 38	2.888	2.824	−0.004	0.146 4	0.028 8
−0.26	−1.452 6	−0.004 69	−0.174 7	−1.664 8	2.977	0.060 72	2.877	2.826	−0.004	0.146 3	0.028 8
−0.24	−1.393 1	−0.005 14	−0.176 0	−1.607 4	2.975	0.066 57	2.866	2.828	−0.004	0.146 1	0.028 8
−0.22	−1.333 6	−0.005 64	−0.177 4	−1.550 2	2.972	0.072 99	2.853	2.831	−0.005	0.146 0	0.028 8
−0.20	−1.274 2	−0.006 18	−0.178 9	−1.493 3	2.970	0.080 02	2.839	2.834	−0.005	0.145 8	0.028 8
−0.18	−1.214 8	−0.006 77	−0.180 6	−1.436 7	2.967	0.087 73	2.824	2.838	−0.006	0.145 6	0.028 8
−0.16	−1.155 5	−0.007 12	−0.182 4	−1.380 4	2.963	0.096 19	2.808	2.840	−0.006	0.145 4	0.028 8
−0.14	−1.096 3	−0.008 13	−0.184 4	−1.324 4	2.960	0.105 5	2.789	2.845	−0.007	0.145 1	0.028 8
−0.12	−1.037 2	−0.008 90	−0.186 6	−1.268 8	2.956	0.115 6	2.770	2.849	−0.008	0.144 9	0.028 7
−0.10	−0.978 1	−0.009 75	−0.189 1	−1.213 6	2.951	0.126 8	2.748	2.854	−0.009	0.144 6	0.028 7
−0.08	−0.919 1	−0.010 68	−0.191 7	−1.158 9	2.947	0.139 0	2.725	2.859	−0.010	0.144 2	0.028 7
−0.06	−0.860 2	−0.011 70	−0.194 6	−1.101 6	2.942	0.152 3	2.699	2.865	−0.011	0.143 9	0.028 7
−0.04	−0.801 4	−0.012 81	−0.197 8	−1.050 9	2.936	0.167 0	2.672	2.871	−0.012	0.143 5	0.028 7
−0.02	−0.742 8	−0.014 02	−0.201 3	−0.997 8	2.930	0.183 1	2.641	2.878	−0.013	0.143 0	0.028 6
−0.00	−0.684 2	−0.015 35	−0.205 2	−0.945 3	2.923	0.200 7	2.608	2.887	−0.014	0.142 5	0.028 6

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.02	−0.625 9	−0.016 81	−0.209 1	−0.893 5	2.916	0.220 0	2.573	2.891	−0.016	0.141 9	0.028 6
+0.04	−0.567 6	−0.018 10	−0.214 0	−0.842 4	2.908	0.241 1	2.534	2.903	−0.018	0.141 3	0.028 5
+0.06	−0.509 6	−0.020 13	−0.219 0	−0.792 1	2.899	0.261 3	2.492	2.913	−0.020	0.140 6	0.028 5
+0.08	−0.451 7	−0.022 03	−0.221 6	−0.742 7	2.889	0.289 7	2.447	2.923	−0.022	0.139 8	0.028 4
+0.10	−0.394 0	−0.024 10	−0.230 6	−0.694 3	2.878	0.317 5	2.398	2.935	−0.024	0.138 9	0.028 4
+0.12	−0.336 6	−0.026 37	−0.237 3	−0.646 8	2.866	0.347 9	2.345	2.947	−0.027	0.137 9	0.028 3
+0.14	−0.279 1	−0.028 83	−0.244 6	−0.600 5	2.854	0.381 2	2.288	2.961	−0.030	0.136 8	0.028 3
+0.16	−0.222 5	−0.031 52	−0.252 5	−0.555 3	2.840	0.417 7	2.227	2.976	−0.033	0.135 5	0.028 2
+0.18	−0.165 8	−0.034 45	−0.261 3	−0.511 4	2.824	0.457 6	2.162	2.992	−0.037	0.134 1	0.028 1
+0.20	−0.109 5	−0.037 65	−0.270 8	−0.468 9	2.807	0.501 4	2.092	3.009	−0.041	0.132 5	0.028 0
+0.22	−0.053 5	−0.041 12	−0.281 3	−0.427 8	2.789	0.549 1	2.017	3.028	−0.046	0.130 7	0.027 9
+0.24	+0.002 0	−0.044 91	−0.292 8	−0.388 2	2.769	0.601 4	1.938	3.049	−0.052	0.128 7	0.027 8
+0.26	+0.057 2	−0.049 02	−0.305 1	−0.350 3	2.746	0.658 5	1.854	3.071	−0.058	0.126 4	0.027 7
+0.28	+0.111 9	−0.053 49	−0.319 2	−0.314 1	2.722	0.720 9	1.765	3.094	−0.065	0.123 8	0.027 5
+0.30	+0.166 1	−0.058 35	−0.334 3	−0.279 7	2.696	0.789 0	1.671	3.119	−0.073	0.120 8	0.027 4
+0.32	+0.219 7	−0.063 62	−0.350 8	−0.247 3	2.667	0.863 3	1.574	3.146	−0.082	0.117 5	0.027 2
+0.34	+0.272 7	−0.069 34	−0.368 9	−0.216 8	2.635	0.944 3	1.472	3.174	−0.093	0.113 7	0.026 9
+0.36	+0.325 1	−0.075 53	−0.388 6	−0.188 4	2.601	1.033	1.367	3.203	−0.104	0.109 5	0.026 7
+0.38	+0.376 7	−0.082 25	−0.410 2	−0.162 1	2.564	1.129	1.258	3.232	−0.117	0.104 6	0.026 4
+0.40	+0.427 6	−0.089 51	−0.433 8	−0.138 1	2.523	1.223	1.148	3.261	−0.132	0.099 2	0.026 1
+0.42	+0.477 6	−0.097 39	−0.459 6	−0.116 2	2.479	1.317	1.037	3.289	−0.148	0.093 1	0.025 8
+0.44	+0.526 8	−0.105 91	−0.487 8	−0.096 6	2.432	1.470	0.926	3.315	−0.167	0.086 3	0.025 4
+0.46	+0.574 9	−0.115 11	−0.518 5	−0.079 2	2.381	1.603	0.816	3.335	−0.187	0.078 6	0.025 0
+0.48	+0.622 0	−0.125 16	−0.552 0	−0.063 9	2.326	1.748	0.708	3.349	−0.210	0.070 2	0.024 6
+0.50	+0.667 9	−0.136 06	−0.588 5	−0.050 8	2.267	1.905	0.605	3.350	−0.231	0.060 9	0.024 1
+0.52	+0.712 7	−0.147 96	−0.628 3	−0.039 7	2.205	2.075	0.506	3.333	−0.261	0.050 9	0.023 6
+0.53	+0.734 6	−0.154 33	−0.649 5	−0.034 9	2.173	2.165	0.459	3.316	−0.275	0.045 6	0.023 4
+0.54	+0.756 1	−0.161 03	−0.671 6	−0.030 5	2.139	2.260	0.414	3.291	−0.290	0.040 2	0.023 1
+0.55	+0.777 3	−0.168 08	−0.691 7	−0.026 6	2.105	2.358	0.371	3.256	−0.304	0.034 8	0.022 8
+0.56	+0.798 2	−0.175 53	−0.718 8	−0.023 1	2.070	2.461	0.330	3.210	−0.318	0.029 3	0.022 6
+0.57	+0.818 7	−0.183 43	−0.743 9	−0.020 0	2.035	2.570	0.291	3.150	−0.333	0.023 9	0.022 3
+0.58	+0.838 9	−0.191 87	−0.770 2	−0.017 3	1.999	2.684	0.254	3.072	−0.346	0.018 8	0.022 1
+0.59	+0.858 7	−0.200 91	−0.797 6	−0.014 9	1.962	2.801	0.220	2.975	−0.359	0.014 0	0.021 8
+0.595	+0.868 5	−0.205 75	−0.811 8	−0.013 8	1.944	2.867	0.201	2.916	−0.364	0.011 9	0.021 7
+0.600	+0.878 2	−0.210 78	−0.826 3	−0.012 9	1.926	2.933	0.188	2.851	−0.369	0.009 9	0.021 6
+0.605	+0.887 7	−0.216 05	−0.841 1	−0.012 0	1.908	3.001	0.173	2.778	−0.374	0.008 3	0.021 6
+0.610	+0.897 2	−0.221 39	−0.856 3	−0.011 1	1.890	3.071	0.158	2.697	−0.377	0.007 0	0.021 5
+0.615	+0.906 4	−0.227 15	−0.871 8	−0.010 4	1.871	3.143	0.141	2.601	−0.380	0.006 1	0.021 5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.620	+0.915 7	−0.233 68	−0.887 7	−0.009 7	1.857	3.221	0.131	2.502	−0.381	0.005 7	0.021 5
+0.625	+0.925 0	−0.240 35	−0.904 0	−0.009 1	1.840	3.303	0.118	2.388	−0.380	0.006 1	0.021 5
					[2. 2]						
+0.12	−0.338 1	−0.026 07	−0.238 3	−0.648 4	2.868	0.346 1	2.349	2.970	−0.029	0.137 4	0.028 3
+0.14	−0.280 9	−0.028 51	−0.245 5	−0.601 9	2.855	0.379 7	2.294	2.988	−0.031	0.135 8	0.028 2
+0.16	−0.223 9	−0.031 16	−0.253 5	−0.556 7	2.841	0.416 0	2.232	3.004	−0.035	0.134 9	0.028 2
+0.18	−0.167 2	−0.034 05	−0.262 2	−0.512 7	2.825	0.455 8	2.168	3.022	−0.039	0.133 4	0.028 1
+0.20	−0.110 8	−0.037 20	−0.271 7	−0.470 0	2.808	0.499 3	2.098	3.042	−0.044	0.131 8	0.028 0
+0.22	−0.054 8	−0.040 62	−0.282 2	−0.428 8	2.789	0.546 9	2.024	3.063	−0.049	0.129 9	0.027 9
+0.24	+0.000 7	−0.044 35	−0.293 6	−0.389 1	2.769	0.597 6	1.945	3.086	−0.055	0.127 7	0.027 8
+0.26	+0.055 9	−0.048 39	−0.306 1	−0.351 0	2.747	0.655 7	1.862	3.110	−0.061	0.125 4	0.027 6
+0.28	+0.110 6	−0.052 79	−0.319 9	−0.314 6	2.722	0.717 7	1.774	3.136	−0.069	0.122 6	0.027 5
+0.30	+0.164 8	−0.057 56	−0.334 9	−0.280 1	2.696	0.785 3	1.681	3.165	−0.077	0.119 5	0.027 3
+0.32	+0.218 4	−0.062 73	−0.351 3	−0.247 4	2.667	0.859 0	1.584	3.195	−0.086	0.116 0	0.027 1
+0.34	+0.271 3	−0.068 32	−0.369 3	−0.216 7	2.635	0.939 5	1.483	3.228	−0.097	0.112 0	0.026 8
+0.36	+0.323 7	−0.074 38	−0.388 9	−0.188 1	2.601	1.027	1.377	3.262	−0.109	0.107 5	0.026 6
+0.38	+0.375 3	−0.080 92	−0.410 4	−0.161 6	2.564	1.122	1.272	3.299	−0.125	0.102 4	0.026 3
+0.40	+0.426 2	−0.087 99	−0.433 9	−0.137 2	2.523	1.225	1.163	3.337	−0.139	0.096 6	0.026 0
+0.42	+0.476 2	−0.095 62	−0.459 5	−0.115 1	2.479	1.337	1.053	3.375	−0.156	0.090 1	0.025 6
+0.44	+0.525 3	−0.103 85	−0.487 4	−0.095 1	2.432	1.458	0.943	3.414	−0.176	0.082 7	0.025 3
+0.46	+0.573 4	−0.112 73	−0.517 9	−0.077 4	2.381	1.590	0.834	3.453	−0.184	0.074 4	0.024 8
+0.48	+0.620 4	−0.122 29	−0.551 1	−0.061 7	2.325	1.732	0.728	3.488	−0.223	0.065 1	0.024 3
+0.50	+0.666 4	−0.132 60	−0.587 2	−0.048 2	2.266	1.884	0.625	3.519	−0.251	0.054 8	0.023 8
+0.52	+0.711 0	−0.143 73	−0.626 5	−0.036 7	2.203	2.049	0.528	3.542	−0.281	0.043 3	0.023 2
+0.54	+0.754 4	−0.155 83	−0.669 3	−0.027 1	2.136	2.226	0.436	3.549	−0.315	0.030 8	0.022 7
+0.56	+0.796 4	−0.168 92	−0.715 7	−0.019 2	2.065	2.417	0.353	3.538	−0.351	0.017 1	0.022 0
+0.57	+0.816 9	−0.175 91	−0.740 4	−0.015 9	2.028	2.518	0.315	3.522	−0.370	0.009 9	0.021 7
+0.58	+0.736 9	−0.183 24	−0.766 1	−0.012 9	1.990	2.623	0.279	3.497	−0.389	0.002 5	0.021 3
+0.59	+0.856 6	−0.190 93	−0.792 8	−0.010 3	1.952	2.733	0.245	3.461	−0.409	−0.005 0	0.021 0
+0.60	+0.876 0	−0.199 05	−0.820 7	−0.008 0	1.911	2.847	0.213	3.412	−0.429	−0.012 6	0.020 6
+0.61	+0.894 9	−0.207 65	−0.849 8	−0.006 0	1.874	2.966	0.184	3.346	−0.448	−0.020 2	0.020 3
+0.62	+0.913 4	−0.216 81	−0.880 1	−0.004 3	1.834	3.092	0.157	3.261	−0.468	−0.027 6	0.020 0
+0.63	+0.931 5	−0.226 65	−0.911 7	−0.002 8	1.794	3.225	0.133	3.151	−0.486	−0.034 5	0.019 7

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.635	+0.940 5	−0.231 90	−0.927 9	−0.002 2	1.771	3.291	0.121	3.085	−0.494	−0.037 7	0.019 6
+0.640	+0.949 3	−0.237 37	−0.944 6	−0.001 6	1.751	3.366	0.111	3.010	−0.502	−0.040 7	0.019 5
+0.645	+0.958 0	−0.243 10	−0.961 6	−0.001 1	1.731	3.441	0.100	2.927	−0.509	−0.043 4	0.019 4
+0.650	+0.966 6	−0.249 14	−0.979 0	−0.000 6	1.715	3.520	0.091	2.833	−0.515	−0.045 7	0.019 3
+0.655	+0.975 1	−0.255 54	−0.996 8	−0.000 2	1.696	3.602	0.081	2.728	−0.519	−0.047 5	0.019 2
+0.660	+0.983 6	−0.262 37	−1.015 1	+0.000 2	1.677	3.688	0.073	2.609	−0.522	−0.048 7	0.019 1
+0.665	+0.991 9	−0.269 71	−1.033 7	+0.000 6	1.659	3.780	0.065	2.476	−0.523	−0.049 0	0.019 1

[2. 3]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.30	+0.161 1	−0.057 17	−0.335 2	−0.280 3	2.695	0.783 6	1.686	3.188	−0.079	0.118 8	0.027 2
+0.32	+0.217 8	−0.062 29	−0.351 6	−0.247 5	2.666	0.857 1	1.589	3.220	−0.088	0.115 2	0.027 0
+0.34	+0.270 7	−0.067 82	−0.369 5	−0.216 7	2.635	0.937 1	1.489	3.255	−0.099	0.111 1	0.026 8
+0.36	+0.323 0	−0.073 81	−0.389 1	−0.188 0	2.601	1.024	1.385	3.292	−0.112	0.106 5	0.026 5
+0.38	+0.374 7	−0.080 28	−0.410 6	−0.161 3	2.563	1.119	1.278	3.332	−0.126	0.101 2	0.026 3
+0.40	+0.425 5	−0.087 25	−0.433 9	−0.136 9	2.523	1.221	1.170	3.375	−0.142	0.095 3	0.025 9
+0.42	+0.475 5	−0.094 76	−0.459 5	−0.114 6	2.479	1.333	1.060	3.419	−0.160	0.088 6	0.025 6
+0.44	+0.524 6	−0.102 85	−0.487 3	−0.094 5	2.431	1.453	0.951	3.466	−0.181	0.080 9	0.025 2
+0.46	+0.572 7	−0.111 55	−0.517 6	−0.076 5	2.380	1.583	0.843	3.513	−0.204	0.072 3	0.024 7
+0.48	+0.619 7	−0.120 89	−0.550 7	−0.060 8	2.325	1.724	0.737	3.561	−0.230	0.062 6	0.024 2
+0.50	+0.665 2	−0.130 93	−0.586 6	−0.047 1	2.267	1.873	0.635	3.603	−0.259	0.051 8	0.023 7
+0.52	+0.709 9	−0.141 70	−0.625 7	−0.035 4	2.204	2.035	0.539	3.645	−0.291	0.039 7	0.023 0
+0.54	+0.753 2	−0.153 28	−0.668 1	−0.025 6	2.136	2.208	0.448	3.680	−0.327	0.026 2	0.022 4
+0.56	+0.795 2	−0.165 74	−0.714 1	−0.017 4	2.061	2.395	0.365	3.705	−0.366	0.011 3	0.021 7
+0.58	+0.835 7	−0.179 18	−0.764 0	−0.010 9	1.988	2.594	0.291	3.712	−0.409	−0.001 9	0.021 0
+0.59	+0.855 4	−0.186 31	−0.790 4	−0.008 1	1.950	2.699	0.258	3.708	−0.431	−0.013 6	0.020 6
+0.60	+0.874 7	−0.193 74	−0.818 0	−0.005 7	1.909	2.807	0.226	3.694	−0.454	−0.022 5	0.020 2
+0.61	+0.893 6	−0.201 52	−0.846 6	−0.003 6	1.868	2.919	0.197	3.672	−0.478	−0.031 7	0.019 8
+0.62	+0.912 1	−0.209 66	−0.876 1	−0.001 8	1.827	3.036	0.170	3.638	−0.503	−0.041 1	0.019 5
+0.63	+0.930 1	−0.218 22	−0.907 3	−0.000 2	1.785	3.157	0.146	3.590	−0.528	−0.050 7	0.019 1
+0.64	+0.947 8	−0.227 28	−0.939 5	+0.001 2	1.743	3.284	0.124	3.524	−0.552	−0.060 3	0.018 7
+0.65	+0.965 0	−0.236 90	−0.973 0	+0.002 3	1.701	3.417	0.104	3.437	−0.577	−0.069 7	0.018 3
+0.66	+0.981 8	−0.247 20	−1.007 9	+0.003 3	1.658	3.558	0.086	3.321	−0.600	−0.078 8	0.017 8

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.665	+0.990 0	−0.252 67	−1.025 9	+0.003 7	1.637	3.631	0.078	3.255	−0.611	−0.083 1	0.017 7
+0.670	+0.998 2	−0.258 37	−1.044 2	+0.004 0	1.616	3.707	0.071	3.179	−0.622	−0.087 2	0.017 5
+0.675	+1.006 2	−0.261 34	−1.062 9	+0.004 4	1.595	3.785	0.063	3.091	−0.632	−0.091 0	0.017 4
+0.680	+1.014 1	−0.270 63	−1.082 1	+0.004 7	1.575	3.868	0.057	2.993	−0.640	−0.094 4	0.017 3
+0.685	+1.022 0	−0.277 29	−1.101 6	+0.004 9	1.554	3.954	0.051	2.882	−0.648	−0.097 3	0.017 2
+0.690	+1.029 7	−0.284 37	−1.121 6	+0.005 2	1.535	4.045	0.044	2.757	−0.654	−0.099 6	0.017 1
+0.695	+1.037 3	−0.291 99	−1.142 1	+0.005 4	1.515	4.142	0.039	2.616	−0.657	−0.101 0	0.017 1
+0.700	+1.044 8	−0.300 26	−1.163 1	+0.005 6	1.197	4.247	0.031	2.156	−0.658	−0.101 3	0.017 0

[2. 4]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.02	−0.628 3	−0.016 45	−0.211 5	−0.897 2	2.916	0.218 6	2.578	2.914	−0.017	0.141 3	0.028 5
+0.04	−0.570 1	−0.018 01	−0.216 1	−0.846 0	2.907	0.239 6	2.539	2.953	−0.019	0.140 6	0.028 5
+0.06	−0.512 0	−0.019 70	−0.221 1	−0.795 6	2.899	0.262 6	2.499	2.964	−0.022	0.139 9	0.028 5
+0.08	−0.454 2	−0.021 55	−0.226 6	−0.746 1	2.889	0.287 8	2.451	2.976	−0.024	0.139 0	0.028 4
+0.10	−0.396 5	−0.023 58	−0.232 6	−0.697 5	2.878	0.315 3	2.406	2.989	−0.027	0.138 0	0.028 3
+0.12	−0.339 1	−0.025 78	−0.239 3	−0.649 9	2.866	0.345 5	2.354	3.002	−0.030	0.136 9	0.028 3
+0.14	−0.281 9	−0.028 18	−0.246 5	−0.603 4	2.853	0.378 5	2.298	3.019	−0.033	0.135 7	0.028 2
+0.16	−0.224 9	−0.030 80	−0.254 1	−0.558 0	2.839	0.414 7	2.237	3.038	−0.037	0.134 3	0.028 1
+0.18	−0.168 3	−0.033 66	−0.263 1	−0.513 9	2.821	0.454 3	2.172	3.055	−0.041	0.132 8	0.028 0
+0.20	−0.112 0	−0.036 76	−0.272 6	−0.471 1	2.807	0.497 6	2.104	3.076	−0.046	0.131 0	0.027 9
+0.22	−0.056 0	−0.040 13	−0.283 0	−0.429 8	2.788	0.544 9	2.031	3.098	−0.051	0.129 0	0.027 8
+0.24	−0.000 5	−0.043 80	−0.294 1	−0.390 0	2.763	0.596 5	1.952	3.123	−0.057	0.126 9	0.027 7
+0.26	+0.054 7	−0.047 78	−0.306 9	−0.351 7	2.746	0.653 0	1.869	3.150	−0.064	0.124 3	0.027 5
+0.28	+0.109 4	−0.052 10	−0.320 6	−0.315 2	2.721	0.714 7	1.782	3.179	−0.072	0.121 4	0.027 4
+0.30	+0.163 5	−0.056 78	−0.335 5	−0.280 5	2.695	0.781 6	1.690	3.210	−0.080	0.118 1	0.027 2
+0.32	+0.217 2	−0.061 85	−0.351 9	−0.247 6	2.666	0.855 1	1.594	3.246	−0.091	0.114 5	0.027 0
+0.34	+0.270 2	−0.067 33	−0.369 8	−0.216 7	2.634	0.934 9	1.494	3.283	−0.102	0.110 3	0.026 7
+0.36	+0.322 5	−0.073 25	−0.389 3	−0.187 9	2.600	1.022	1.425	3.324	−0.115	0.105 5	0.026 5
+0.38	+0.374 1	−0.079 63	−0.410 7	−0.161 1	2.562	1.116	1.285	3.367	−0.129	0.100 1	0.026 2
+0.40	+0.425 0	−0.086 51	−0.434 0	−0.136 5	2.522	1.218	1.177	3.414	−0.146	0.094 0	0.025 9
+0.42	+0.474 9	−0.093 91	−0.459 5	−0.111 0	2.478	1.329	1.068	3.461	−0.165	0.087 0	0.025 5
+0.44	+0.524 0	−0.101 86	−0.487 2	−0.093 8	2.430	1.448	0.959	3.517	−0.186	0.079 2	0.025 1
+0.46	+0.572 1	−0.110 39	−0.517 1	−0.075 7	2.379	1.575	0.851	3.568	−0.210	0.070 2	0.024 6
+0.48	+0.619 1	−0.119 53	−0.550 3	−0.059 7	2.323	1.716	0.746	3.631	−0.236	0.060 3	0.024 1
+0.50	+0.664 9	−0.129 30	−0.586 1	−0.045 8	2.263	1.865	0.645	3.691	−0.266	0.048 9	0.023 5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.52	+0.709 6	−0.139 75	−0.625 0	−0.033 9	2.200	2.025	0.548	3.752	−0.300	0.036 1	0.022 9
+0.54	+0.752 9	−0.150 90	−0.667 2	−0.023 8	2.131	2.196	0.459	3.813	−0.338	0.021 8	0.022 2
+0.56	+0.791 8	−0.162 80	−0.712 9	−0.015 4	2.059	2.378	0.376	3.871	−0.382	0.005 8	0.021 4
+0.58	+0.835 2	−0.175 19	−0.762 3	−0.008 7	1.983	2.570	0.302	3.924	−0.427	−0.011 9	0.020 7
+0.60	+0.871 0	−0.189 02	−0.815 8	−0.003 3	1.903	2.775	0.238	3.969	−0.478	−0.031 5	0.019 8
+0.62	+0.911 2	−0.203 98	−0.873 1	+0.000 9	1.802	2.991	0.182	4.005	−0.530	−0.051 7	0.019 0
+0.64	+0.946 8	−0.218 97	−0.935 5	+0.004 1	1.733	3.219	0.136	4.011	−0.594	−0.076 3	0.018 0
+0.65	+0.963 9	−0.227 15	−0.968 3	+0.005 3	1.689	3.339	0.116	4.005	−0.626	−0.088 7	0.017 5
+0.66	+0.980 5	−0.235 65	−1.002 3	+0.006 1	1.644	3.460	0.098	3.989	−0.658	−0.101 4	0.017 0
+0.67	+0.996 7	−0.244 51	−1.037 5	+0.007 3	1.599	3.587	0.083	3.962	−0.692	−0.114 5	0.016 6
+0.68	+1.012 5	−0.253 77	−1.074 0	+0.008 0	1.553	3.716	0.069	3.921	−0.726	−0.127 9	0.016 2
+0.69	+1.027 8	−0.263 49	−1.111 8	+0.008 7	1.507	3.851	0.057	3.862	−0.760	−0.141 4	0.015 7
+0.70	+1.042 6	−0.273 74	−1.151 0	+0.009 2	1.459	3.991	0.046	3.780	−0.795	−0.155 0	0.015 2
+0.71	+1.057 0	−0.284 61	−1.191 7	+0.009 6	1.417	4.140	0.037	3.676	−0.829	−0.168 4	0.014 7
+0.72	+1.071 0	−0.296 33	−1.233 9	+0.010 0	1.372	4.295	0.029	3.534	−0.861	−0.181 3	0.014 3
+0.725	+1.077 8	−0.302 51	−1.255 5	+0.010 1	1.349	4.377	0.026	3.448	−0.877	−0.187 5	0.014 1
+0.730	+1.084 5	−0.309 04	−1.277 6	+0.010 2	1.327	4.462	0.023	3.352	−0.892	−0.193 4	0.013 9
+0.735	+1.091 1	−0.315 87	−1.300 2	+0.010 3	1.305	4.551	0.020	3.243	−0.905	−0.198 8	0.013 8
+0.740	+1.097 5	−0.323 10	−1.323 2	+0.010 4	1.284	4.644	0.017	3.118	−0.918	−0.203 7	0.013 7
+0.745	+1.103 9	−0.330 79	−1.346 6	+0.010 5	1.264	4.741	0.015	2.977	−0.928	−0.207 9	0.013 6
+0.750	+1.110 2	−0.339 07	−1.370 6	+0.010 6	1.244	4.850	0.013	2.816	−0.936	−0.211 2	0.013 5
+0.755	+1.116 4	−0.348 08	−1.395 2	+0.010 6	1.225	4.966	0.011	2.634	−0.942	−0.213 3	0.013 4
+0.760	+1.122 4	−0.358 02	−1.420 3	+0.010 7	1.208	5.094	0.009	2.425	−0.942	−0.213 5	0.013 4

[2. 5]

+0.40	+0.424 8	−0.086 33	−0.434 0	−0.136 4	2.522	1.217	1.179	3.423	−0.147	0.093 7	0.025 8
+0.42	+0.474 8	−0.093 69	−0.459 1	−0.113 9	2.478	1.328	1.069	3.475	−0.165	0.086 7	0.025 5
+0.44	+0.523 9	−0.101 61	−0.487 1	−0.093 6	2.430	1.417	0.961	3.530	−0.187	0.078 8	0.025 0
+0.46	+0.572 0	−0.110 10	−0.517 3	−0.075 4	2.378	1.576	0.854	3.588	−0.211	0.069 8	0.024 6
+0.48	+0.619 0	−0.119 18	−0.550 2	−0.059 4	2.323	1.711	0.749	3.650	−0.238	0.059 6	0.024 1
+0.50	+0.664 8	−0.128 90	−0.586 0	−0.045 5	2.263	1.863	0.647	3.714	−0.268	0.048 1	0.023 5
+0.52	+0.709 5	−0.139 26	−0.624 8	−0.033 5	2.199	2.022	0.551	3.780	−0.303	0.035 2	0.022 9
+0.54	+0.752 7	−0.150 51	−0.666 9	−0.023 1	2.131	2.192	0.461	3.847	−0.341	0.020 7	0.022 2
+0.56	+0.794 6	−0.162 07	−0.712 6	−0.015 0	2.058	2.373	0.379	3.911	−0.393	2.004 5	0.021 4
+0.58	+0.835 0	−0.174 57	−0.761 9	−0.008 1	1.982	2.565	0.305	3.979	−0.431	−0.013 6	0.020 6
+0.60	+0.873 8	−0.187 86	−0.815 2	−0.002 7	1.902	2.767	0.241	4.039	−0.484	−0.033 7	0.019 8

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.62	+0.911 0	−0.201 99	−0.872 6	+0.001 5	1.818	2.980	0.185	4.092	−0.541	−0.055 8	0.018 9
+0.64	+0.946 5	−0.217 02	−0.931 5	+0.004 8	1.731	3.204	0.139	4.132	−0.604	−0.080 1	0.017 8
+0.66	+0.980 2	−0.233 04	−1.000 9	+0.007 2	1.641	3.439	0.101	4.154	−0.671	−0.106 4	0.016 9
+0.68	+1.012 1	−0.250 18	−1.072 1	+0.008 9	1.549	3.684	0.071	4.148	−0.743	−0.134 8	0.015 9
+0.69	+1.027 4	−0.259 24	−1.109 6	+0.009 5	1.503	3.813	0.059	4.130	−0.781	−0.149 5	0.015 4
+0.70	+1.042 2	−0.268 66	−1.148 3	+0.010 1	1.457	3.945	0.049	4.100	−0.819	−0.164 6	0.014 8
+0.71	+1.056 5	−0.278 50	−1.188 5	+0.010 5	1.410	4.080	0.040	4.054	−0.859	−0.180 3	0.014 3
+0.72	+1.070 4	−0.288 81	−1.230 0	+0.010 9	1.363	4.220	0.032	3.989	−0.898	−0.196 0	0.013 9
+0.73	+1.083 8	−0.299 69	−1.272 9	+0.011 2	1.317	4.365	0.025	3.901	−0.938	−0.211 7	0.013 5
+0.74	+1.096 7	−0.311 24	−1.317 3	+0.011 4	1.271	4.518	0.021	3.783	−0.977	−0.227 2	0.013 0
+0.745	+1.103 0	−0.317 32	−1.340 1	+0.011 5	1.249	4.598	0.018	3.711	−0.996	−0.234 8	0.012 8
+0.750	+1.109 2	−0.323 64	−1.363 3	+0.011 6	1.226	4.681	0.016	3.628	−1.014	−0.242 2	0.012 5
+0.755	+1.115 3	−0.330 23	−1.386 9	+0.011 6	1.204	4.766	0.014	3.534	−1.032	−0.249 4	0.012 3
+0.760	+1.121 2	−0.337 11	−1.411 0	+0.011 7	1.182	4.855	0.012	3.426	−1.049	−0.256 1	0.012 2
+0.765	+1.127 1	−0.344 42	−1.435 5	+0.011 8	1.161	4.948	0.010	3.305	−1.061	−0.262 4	0.012 0
+0.770	+1.132 8	−0.352 14	−1.460 4	+0.011 8	1.140	5.048	0.009	3.166	−1.078	−0.268 1	0.011 9
+0.775	+1.138 5	−0.360 10	−1.486 0	+0.011 9	1.120	5.154	0.007	3.008	−1.091	−0.273 0	0.011 8
+0.780	+1.144 0	−0.369 33	−1.512 0	+0.011 9	1.101	5.269	0.006	3.828	−1.100	−0.276 8	0.011 7
+0.785	+1.149 5	−0.379 11	−1.538 7	+0.011 9	1.082	5.391	0.005	2.621	−1.105	−0.278 9	0.011 7
+0.790	+1.154 9	−0.390 02	−1.566 0	+0.012 0	1.065	5.536	0.001	2.386	−1.105	−0.279 0	0.011 7

[2. 6]

+0.40	+0.424 7	−0.086 28	−0.431 0	−0.136 3	2.522	1.217	1.177	3.426	−0.147	0.093 6	0.025 8
+0.42	+0.474 7	−0.093 64	−0.459 4	−0.113 9	2.478	1.327	1.079	3.478	−0.166	0.086 6	0.025 5
+0.44	+0.523 8	−0.101 55	−0.487 2	−0.093 5	2.429	1.461	0.961	3.534	−0.182	0.078 6	0.025 1
+0.46	+0.571 8	−0.110 02	−0.517 3	−0.075 4	2.379	1.575	0.851	3.592	−0.211	0.069 6	0.024 6
+0.48	+0.618 8	−0.119 09	−0.550 2	−0.059 4	2.323	1.711	0.749	3.654	−0.238	0.059 4	0.024 0
+0.50	+0.664 7	−0.128 79	−0.585 9	−0.045 4	2.263	1.862	0.648	3.719	−0.269	0.047 9	0.023 5
+0.52	+0.709 3	−0.139 13	−0.624 8	−0.033 4	2.199	2.021	0.552	3.786	−0.304	0.035 0	0.022 8
+0.54	+0.752 6	−0.150 15	−0.666 9	−0.023 3	2.131	2.191	0.462	3.855	−0.342	0.020 4	0.022 2
+0.56	+0.791 5	−0.161 87	−0.712 5	−0.014 9	2.059	2.371	0.380	3.924	−0.384	0.004 1	0.021 4
+0.58	+0.834 9	−0.174 33	−0.761 8	−0.008 0	1.982	2.562	0.306	3.992	−0.432	−0.014 0	0.020 6
+0.60	+0.873 7	−0.187 56	−0.815 0	−0.002 6	1.902	2.761	0.241	4.056	−0.485	−0.034 2	0.019 7
+0.62	+0.910 9	−0.201 61	−0.872 4	+0.001 7	1.818	2.976	0.186	3.115	−0.543	−0.056 5	0.018 8
+0.64	+0.946 3	−0.216 52	−0.934 1	+0.004 9	1.731	3.200	0.140	4.162	−0.606	−0.080 9	0.017 8
+0.66	+0.980 1	−0.232 38	−1.000 4	+0.007 3	1.642	3.433	0.102	4.195	−0.674	−0.107 6	0.016 8
+0.68	+1.011 9	−0.249 30	−1.071 5	+0.009 1	1.549	3.676	0.072	4.201	−0.747	−0.136 3	0.015 9

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
+0.69	+1.027 2	−0.258 19	−1.108 9	+0.009 7	1.503	3.803	0.060	4.197	−0.787	−0.151 5	0.015 3
+0.70	+1.042 0	−0.267 43	−1.147 6	+0.010 3	1.456	3.933	0.050	4.178	−0.825	−0.167 1	0.014 7
+0.71	+1.056 3	−0.277 03	−1.187 6	+0.010 7	1.409	4.065	0.041	4.148	−0.862	−0.181 7	0.014 3
+0.72	+1.070 1	−0.287 05	−1.228 9	+0.011 1	1.362	4.200	0.033	4.101	−0.903	−0.197 8	0.013 9
+0.73	+1.083 5	−0.297 54	−1.271 6	+0.011 1	1.316	4.312	0.026	4.035	−0.918	−0.215 8	0.013 3
+0.74	+1.096 4	−0.308 60	−1.315 7	+0.011 6	1.269	4.490	0.021	3.945	−0.989	−0.232 3	0.012 8
+0.75	+1.108 9	−0.320 35	−1.361 1	+0.011 8	1.221	4.614	0.016	3.825	−1.030	−0.248 6	0.012 4
+0.755	+1.115 0	−0.326 53	−1.384 8	+0.011 9	1.201	4.721	0.014	3.751	−1.050	−0.256 6	0.012 2
+0.760	+1.120 9	−0.332 95	−1.408 6	+0.012 0	1.179	4.807	0.012	3.667	−1.069	−0.261 4	0.012 0
+0.765	+1.126 7	−0.339 65	−1.432 9	+0.012 0	1.157	4.893	0.011	3.571	−1.088	−0.271 8	0.011 8
+0.770	+1.132 5	−0.346 68	−1.457 6	+0.012 1	1.135	4.982	0.009	3.461	−1.105	−0.279 0	0.011 7
+0.775	+1.138 1	−0.354 07	−1.482 7	+0.012 1	1.113	5.077	0.008	3.335	−1.122	−0.285 6	0.011 5
+0.780	+1.143 6	−0.361 93	−1.508 1	+0.012 2	1.093	5.178	0.007	3.193	−1.137	−0.291 6	0.011 3
+0.785	+1.149 0	−0.370 33	−1.534 5	+0.012 2	1.073	5.286	0.006	3.031	−1.149	−0.296 8	0.011 2
+0.790	+1.154 3	−0.379 42	−1.561 3	+0.012 2	1.054	5.403	0.005	2.816	−1.159	−0.300 8	0.011 1
+0.795	+1.159 6	−0.389 39	−1.588 6	+0.012 2	1.036	5.532	0.004	2.634	−1.165	−0.303 2	0.011 0
+0.800	+1.164 7	−0.400 55	−1.616 6	+0.012 3	1.019	5.678	0.003	2.390	−1.166	−0.303 3	0.011 0

[2. 7]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
−0.54	−2.292 0	−0.001 26	−0.168 2	−2.491 4	2.993	0.016 61	2.966	2.857	−0.001	−0.147 3	0.028 9
−0.52	−2.232 1	−0.001 39	−0.168 6	−2.432 1	2.993	0.018 18	2.963	2.858	−0.001	−0.147 2	0.028 9
−0.50	−2.172 3	−0.001 52	−0.168 9	−2.372 8	2.992	0.019 97	2.959	2.859	−0.001	−0.147 2	0.028 9
−0.48	−2.112 4	−0.001 67	−0.169 4	−2.313 7	2.991	0.021 90	2.955	2.860	−0.002	−0.147 1	0.028 9
−0.46	−2.052 6	−0.001 83	−0.169 8	−2.254 6	2.991	0.024 01	2.951	2.862	−0.002	−0.147 1	0.028 9
−0.44	−1.992 8	−0.002 00	−0.170 3	−2.195 7	2.990	0.026 33	2.947	2.863	−0.002	−0.147 0	0.028 9
−0.42	−1.933 0	−0.002 20	−0.170 9	−2.136 8	2.989	0.028 86	2.941	2.864	−0.002	−0.147 0	0.028 9
−0.40	−1.873 3	−0.002 41	−0.171 5	−2.078 0	2.988	0.031 65	2.936	2.865	−0.002	−0.146 9	0.028 9
−0.38	−1.813 5	−0.002 64	−0.172 1	−2.019 4	2.987	0.034 70	2.930	2.867	−0.003	−0.146 8	0.028 9
−0.36	−1.753 8	−0.002 89	−0.172 9	−1.960 8	2.985	0.038 04	2.923	2.868	−0.003	−0.146 7	0.028 8
−0.34	−1.694 1	−0.003 17	−0.173 7	−1.902 4	2.984	0.041 71	2.916	2.870	−0.003	−0.146 6	0.028 8
−0.32	−1.634 4	−0.003 48	−0.174 5	−1.844 2	2.982	0.045 73	2.908	2.872	−0.003	−0.146 5	0.028 8
−0.30	−1.571 8	−0.003 81	−0.175 5	−1.786 1	2.981	0.050 19	2.899	2.874	−0.004	−0.146 4	0.028 8
−0.28	−1.515 2	−0.004 18	−0.176 5	−1.728 2	2.979	0.054 98	2.890	2.876	−0.004	−0.146 3	0.028 8
−0.26	−1.455 7	−0.004 58	−0.177 7	−1.670 5	2.977	0.060 27	2.879	2.878	−0.004	−0.146 1	0.028 8
−0.24	−1.396 1	−0.005 01	−0.178 9	−1.613 1	2.975	0.066 08	2.868	2.881	−0.005	−0.145 9	0.028 8
−0.22	−1.336 7	−0.005 50	−0.180 3	−1.555 8	2.972	0.072 45	2.855	2.885	−0.005	−0.145 8	0.028 8
−0.20	−1.277 3	−0.006 02	−0.181 8	−1.498 9	2.969	0.079 44	2.842	2.888	−0.006	−0.145 5	0.028 8

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
−0.18	−1.217 9	−0.006 60	−0.183 5	−1.442 2	2.966	0.087 09	2.827	2.892	−0.007	−0.145 3	0.028 8
−0.16	−1.158 6	−0.007 23	−0.185 3	−1.385 8	2.963	0.095 48	2.810	2.896	−0.007	−0.145 1	0.028 8
−0.14	−1.099 4	−0.007 92	−0.187 3	−1.329 8	2.960	0.104 7	2.793	2.900	−0.008	−0.144 8	0.028 7
−0.12	−1.040 2	−0.008 67	−0.189 5	−1.274 1	2.956	0.114 7	2.773	2.905	−0.009	−0.144 5	0.028 7
−0.10	−0.981 1	−0.009 50	−0.191 9	−1.218 9	2.951	0.125 8	2.752	2.910	−0.010	−0.144 2	0.028 7
−0.08	−0.922 2	−0.010 40	−0.194 6	−1.164 1	2.947	0.137 9	2.729	2.916	−0.011	−0.143 8	0.028 7
−0.06	−0.863 3	−0.011 39	−0.197 4	−1.109 7	2.942	0.151 2	2.704	2.923	−0.012	−0.143 4	0.028 7
−0.04	−0.804 5	−0.012 47	−0.200 6	−1.055 9	2.936	0.165 7	2.682	2.930	−0.013	−0.142 9	0.028 6
−0.02	−0.745 9	−0.013 66	−0.204 1	−1.002 7	2.930	0.181 6	2.647	2.937	−0.015	−0.141 4	0.028 6
−0.00	−0.687 3	−0.014 95	−0.207 9	−0.950 1	2.923	0.199 1	2.615	2.946	−0.016	−0.141 8	0.028 6
+0.02	−0.628 9	−0.016 36	−0.212 1	−0.898 1	2.916	0.218 2	2.579	2.955	−0.018	−0.141 2	0.028 5
+0.04	−0.570 7	−0.017 91	−0.216 6	−0.846 9	2.907	0.239 2	2.541	2.965	−0.020	−0.140 5	0.028 5
+0.06	−0.512 6	−0.019 59	−0.221 6	−0.796 5	2.898	0.262 1	2.500	2.977	−0.022	−0.139 7	0.028 4
+0.08	−0.454 8	−0.021 43	−0.227 1	−0.746 9	2.889	0.287 3	2.456	2.989	−0.024	−0.138 8	0.028 4
+0.10	−0.397 1	−0.023 44	−0.233 1	−0.698 3	2.878	0.314 8	2.407	3.003	−0.027	−0.137 8	0.028 3
+0.12	−0.339 7	−0.025 63	−0.239 7	−0.650 6	2.866	0.344 9	2.356	3.017	−0.030	−0.136 7	0.028 3
+0.14	−0.282 5	−0.028 02	−0.247 0	−0.604 1	2.853	0.377 9	2.300	3.034	−0.033	−0.135 4	0.028 2
+0.16	−0.225 6	−0.030 62	−0.254 9	−0.558 7	2.839	0.414 0	2.240	3.051	−0.037	−0.134 0	0.028 1
+0.18	−0.168 9	−0.033 45	−0.263 5	−0.514 5	2.824	0.453 1	2.176	3.071	−0.042	−0.132 4	0.028 0
+0.20	−0.112 6	−0.036 53	−0.273 0	−0.471 7	2.807	0.496 6	2.107	3.092	−0.047	−0.130 6	0.027 9
+0.22	−0.056 7	−0.039 88	−0.283 4	−0.430 3	2.788	0.543 8	2.034	3.116	−0.052	0.128 6	0.027 8
+0.24	+0.001 1	−0.043 52	−0.294 8	−0.390 4	2.768	0.595 1	1.956	3.141	−0.058	0.126 3	0.027 7
+0.26	+0.054 0	−0.047 47	−0.307 3	−0.352 1	2.746	0.651 7	1.873	3.170	−0.065	0.123 7	0.027 5
+0.28	+0.108 7	−0.051 75	−0.320 9	−0.315 4	2.721	0.713 1	1.786	3.199	−0.073	0.120 8	0.027 3
+0.30	+0.162 8	−0.056 39	−0.335 8	−0.280 6	2.696	0.779 9	1.695	3.233	−0.082	0.117 5	0.027 2
+0.32	+0.216 1	−0.061 44	−0.352 1	−0.247 7	2.666	0.852 9	1.599	3.269	−0.092	0.113 7	0.026 9
+0.34	+0.269 1	−0.066 84	−0.370 0	−0.216 7	2.636	0.932 7	1.501	3.309	−0.104	0.109 4	0.026 7
+0.36	+0.321 8	−0.072 69	−0.389 5	−0.187 7	2.600	1.019	1.397	3.352	−0.117	0.104 6	0.026 4
+0.38	+0.373 1	−0.079 00	−0.410 8	−0.160 8	2.565	1.113	1.291	3.402	−0.132	0.099 0	0.026 1
+0.40	+0.424 3	−0.085 79	−0.434 1	−0.136 1	2.522	1.214	1.184	3.451	−0.149	0.092 7	0.025 8
+0.42	+0.474 2	−0.093 08	−0.459 4	−0.113 5	2.478	1.321	1.073	3.507	−0.168	0.085 6	0.025 4
+0.44	+0.523 3	−0.100 90	−0.487 1	−0.093 1	2.430	1.443	0.967	3.569	−0.190	0.077 5	0.025 0
+0.46	+0.571 4	−0.109 27	−0.517 2	−0.074 8	2.378	1.571	0.860	3.631	−0.215	0.068 3	0.024 5
+0.48	+0.618 1	−0.118 21	−0.549 9	−0.058 6	2.322	1.709	0.755	3.700	−0.242	0.057 9	0.024 0

[3. 1]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
−0.54	−2.277 5	−0.001 42	−0.182 9	−2.463 0	2.993	0.018 36	2.963	2.808
−0.52	−2.217 7	−0.001 56	−0.183 2	−2.403 8	2.992	0.020 11	2.959	2.809
−0.50	−2.157 8	−0.001 71	−0.183 7	−2.341 7	2.991	0.022 07	2.955	2.810
−0.48	−2.098 0	−0.001 87	−0.184 1	−2.285 7	2.991	0.024 21	2.951	2.811
−0.46	−2.038 2	−0.002 05	−0.184 6	−2.226 7	2.990	0.026 53	2.946	2.812
−0.44	−1.978 4	−0.002 25	−0.185 2	−2.167 8	2.989	0.029 10	2.941	2.813
−0.42	−1.918 7	−0.002 47	−0.185 8	−2.109 1	2.988	0.031 90	2.935	2.814
−0.40	−1.858 9	−0.002 71	−0.186 5	−2.050 1	2.987	0.034 98	2.929	2.815
−0.38	−1.799 2	−0.002 97	−0.187 2	−1.992 0	2.985	0.038 30	2.922	2.817
−0.36	−1.739 5	−0.003 25	−0.188 0	−1.933 6	2.984	0.042 05	2.915	2.819
−0.34	−1.679 9	−0.003 56	−0.188 9	−1.875 4	2.982	0.046 10	2.907	2.820
−0.32	−1.620 3	−0.003 91	−0.189 8	−1.817 3	2.981	0.050 51	2.898	2.822
−0.30	−1.560 7	−0.004 28	−0.190 9	−1.759 5	2.979	0.055 41	2.888	2.824
−0.28	−1.501 2	−0.004 69	−0.192 1	−1.701 9	2.977	0.060 75	2.877	2.827
−0.26	−1.441 6	−0.005 14	−0.193 3	−1.641 4	2.975	0.066 61	2.866	2.829
−0.24	−1.382 2	−0.005 64	−0.194 7	−1.587 3	2.972	0.073 03	2.853	2.832
−0.22	−1.322 8	−0.006 18	−0.196 3	−1.530 4	2.969	0.080 06	2.839	2.835
−0.20	−1.263 4	−0.006 77	−0.197 9	−1.473 7	2.966	0.087 78	2.824	2.838
−0.18	−1.204 1	−0.007 42	−0.199 8	−1.417 1	2.963	0.096 23	2.807	2.842
−0.16	−1.144 9	−0.008 12	−0.201 8	−1.361 5	2.960	0.105 5	2.789	2.846
−0.14	−1.085 7	−0.008 90	−0.204 0	−1.305 9	2.956	0.115 7	2.769	2.851
−0.12	−1.026 7	−0.009 75	−0.206 4	−1.250 7	2.951	0.126 8	2.747	2.856
−0.10	−0.967 7	−0.010 68	−0.209 1	−1.196 0	2.946	0.139 0	2.724	2.861
−0.08	−0.908 8	−0.011 69	−0.212 0	−1.141 8	2.941	0.152 4	2.698	2.867
−0.06	−0.850 1	−0.012 81	−0.215 2	−1.088 1	2.936	0.167 0	2.652	2.873
−0.04	−0.791 1	−0.014 02	−0.218 7	−1.035 0	2.929	0.183 1	2.640	2.881
−0.02	−0.732 9	−0.015 35	−0.222 5	−0.982 6	2.923	0.200 7	2.607	2.888
−0.00	−0.674 5	−0.016 81	−0.226 7	−0.930 8	2.915	0.219 4	2.571	2.891
+0.02	−0.616 3	−0.018 39	−0.231 3	−0.879 8	2.907	0.241 0	2.532	2.905
+0.04	−0.558 3	−0.020 13	−0.236 1	−0.829 5	2.898	0.264 2	2.491	2.916
+0.06	−0.500 4	−0.022 02	−0.241 9	−0.780 2	2.888	0.289 5	2.445	2.927
+0.08	−0.442 8	−0.024 09	−0.248 0	−0.731 8	2.877	0.317 2	2.396	2.940
+0.10	−0.385 4	−0.026 34	−0.254 6	−0.684 4	2.865	0.345 3	2.343	2.953

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
+0.12	−0.328 2	−0.028 80	−0.261 9	−0.638 1	2.852	0.370 7	2.286	2.967
+0.14	−0.271 3	−0.031 18	−0.269 9	−0.593 0	2.837	0.417 0	2.225	2.983
+0.16	−0.214 7	−0.034 40	−0.278 6	−0.549 1	2.822	0.456 8	2.159	3.000
+0.18	−0.158 1	−0.037 58	−0.288 1	−0.506 6	2.805	0.500 2	2.089	3.018
+0.20	−0.102 5	−0.041 03	−0.298 6	−0.465 6	2.786	0.517 6	2.015	3.039
+0.22	−0.047 0	−0.044 79	−0.310 1	−0.426 0	2.765	0.599 5	1.935	3.061
+0.24	+0.008 1	−0.048 87	−0.322 6	−0.388 2	2.743	0.656 0	1.851	3.084
+0.26	+0.062 7	−0.053 30	−0.336 3	−0.352 0	2.718	0.717 8	1.763	3.110
+0.28	+0.116 8	−0.058 11	−0.351 4	−0.317 7	2.691	0.785 1	1.669	3.136
+0.30	+0.170 4	−0.063 32	−0.367 8	−0.285 3	2.662	0.858 1	1.572	3.168
+0.32	+0.223 3	−0.068 96	−0.385 7	−0.254 8	2.630	0.938 1	1.471	3.199
+0.34	+0.275 5	−0.075 05	−0.405 4	−0.226 5	2.595	1.024	1.366	3.234
+0.36	+0.327 1	−0.081 63	−0.426 8	−0.200 2	2.557	1.119	1.259	3.270
+0.38	+0.377 8	−0.088 74	−0.450 2	−0.177 0	2.516	1.221	1.153	3.312
+0.40	+0.427 7	−0.096 39	−0.475 7	−0.155 0	2.472	1.331	1.043	3.351
+0.42	+0.476 7	−0.104 63	−0.503 5	−0.135 3	2.424	1.450	0.933	3.389
+0.44	+0.524 6	−0.113 51	−0.533 7	−0.117 7	2.372	1.578	0.825	3.426
+0.46	+0.571 5	−0.123 07	−0.566 7	−0.102 3	2.317	1.716	0.719	3.461
+0.48	+0.617 3	−0.133 36	−0.602 5	−0.088 9	2.257	1.865	0.617	3.491
+0.50	+0.661 8	−0.144 45	−0.641 3	−0.077 5	2.191	2.024	0.521	3.513
+0.52	+0.705 1	−0.156 13	−0.683 5	−0.068 0	2.128	2.195	0.432	3.523
+0.53	+0.726 2	−0.162 79	−0.705 9	−0.063 9	2.093	2.285	0.390	3.520
+0.54	+0.746 9	−0.169 43	−0.729 2	−0.060 2	2.057	2.379	0.350	3.512
+0.55	+0.767 3	−0.176 36	−0.753 5	−0.056 9	2.021	2.476	0.312	3.496
+0.56	+0.787 3	−0.183 60	−0.778 8	−0.054 0	1.983	2.576	0.277	3.472
+0.57	+0.807 0	−0.191 21	−0.805 1	−0.051 4	1.946	2.680	0.244	3.438
+0.58	+0.826 2	−0.199 22	−0.832 4	−0.049 1	1.908	2.789	0.213	3.390
+0.59	+0.845 1	−0.207 69	−0.860 8	−0.047 1	1.869	2.903	0.181	3.326
+0.60	+0.863 6	−0.216 70	−0.890 5	−0.045 4	1.831	3.022	0.158	3.241
+0.61	+0.881 8	−0.226 37	−0.921 3	−0.043 9	1.792	3.149	0.131	3.138
+0.615	+0.890 7	−0.231 49	−0.937 2	−0.043 3	1.773	3.215	0.122	3.075
+0.620	+0.899 5	−0.236 84	−0.953 5	−0.042 7	1.754	3.283	0.112	3.001
+0.625	+0.908 2	−0.242 43	−0.970 0	−0.042 2	1.735	3.354	0.101	2.924
+0.630	+0.916 8	−0.248 33	−0.987 0	−0.041 7	1.716	3.429	0.092	2.824
+0.635	+0.925 3	−0.254 56	−1.004 3	−0.041 2	1.698	3.507	0.084	2.733
+0.640	+0.933 8	−0.261 18	−1.022 1	−0.040 8	1.680	3.589	0.074	2.619
+0.645	+0.941 8	−0.268 29	−1.040 2	−0.040 5	1.664	3.673	0.067	2.491

[3. 2]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
+0.28	+0.116 1	−0.057 71	−0.351 7	−0.317 7	2.691	0.783 1	1.673	3.159
+0.30	+0.169 7	−0.062 86	−0.368 0	−0.285 4	2.662	0.856 1	1.577	3.193
+0.32	+0.222 6	−0.068 41	−0.386 0	−0.254 8	2.630	0.935 5	1.476	3.227
+0.34	+0.274 8	−0.074 47	−0.405 5	−0.226 3	2.595	1.022	1.372	3.263
+0.36	+0.326 3	−0.080 98	−0.426 9	−0.200 0	2.557	1.115	1.266	3.302
+0.38	+0.377 1	−0.087 99	−0.450 2	−0.175 7	2.516	1.216	1.157	3.343
+0.40	+0.426 9	−0.095 51	−0.475 6	−0.153 7	2.472	1.326	1.018	3.386
+0.42	+0.475 9	−00103 66	−0.503 3	−0.133 8	2.424	1.441	0.939	3.430
+0.44	+0.523 8	−0.112 39	−0.533 4	−0.116 1	2.372	1.571	0.831	3.474
+0.46	+0.570 7	−0.121 76	−0.566 2	−0.100 5	2.317	1.708	0.726	3.518
+0.48	+0.616 5	−0.131 82	−0.601 8	−0.087 0	2.257	1.855	0.625	3.561
+0.50	+0.661 0	−0.142 62	−0.640 4	−0.075 5	2.194	2.012	0.530	3.598
+0.52	+0.704 2	−0.154 21	−0.682 3	−0.065 8	2.127	2.180	0.441	3.628
+0.54	+0.746 0	−0.166 68	−0.727 7	−0.057 8	2.056	2.359	0.360	3.646
+0.56	+0.786 4	−0.180 13	−0.776 7	−0.051 3	1.982	2.550	0.287	3.646
+0.57	+0.806 0	−0.187 27	−0.802 7	−0.048 6	1.944	2.650	0.251	3.635
+0.58	+0.825 2	−0.194 72	−0.829 7	−0.046 2	1.904	2.754	0.223	3.617
+0.59	+0.844 1	−0.202 51	−0.857 8	−0.044 1	1.865	2.861	0.194	3.588
+0.60	+0.862 5	−0.210 68	−0.887 0	−0.042 3	1.825	2.973	0.168	3.547
+0.61	+0.880 6	−0.219 29	−0.917 3	−0.040 8	1.784	3.089	0.144	3.491
+0.62	+0.898 2	−0.228 41	−0.948 8	−0.039 1	1.744	3.211	0.123	3.416
+0.63	+0.915 4	−0.238 13	−0.981 5	−0.038 3	1.703	3.339	0.102	3.319
+0.64	+0.932 3	−0.248 58	−1.015 6	−0.037 4	1.662	3.474	0.085	3.194
+0.645	+0.940 5	−0.254 11	−1.033 1	−0.037 0	1.642	3.545	0.077	3.119
+0.650	+0.948 7	−0.259 96	−1.051 1	−0.036 6	1.623	3.620	0.070	3.035
+0.655	+0.956 8	−0.266 08	−1.069 1	−0.036 3	1.603	3.697	0.063	2.910
+0.660	+0.964 7	−0.272 56	−1.088 1	−0.036 0	1.584	3.778	0.056	2.833
+0.665	+0.972 6	−0.279 44	−1.107 2	−0.035 7	1.566	3.864	0.050	2.713
+0.670	+0.980 4	−0.286 83	−1.126 7	−0.035 5	1.548	3.956	0.044	2.578
+0.675	+0.988 1	−0.294 83	−1.146 7	−0.035 3	1.531	4.054	0.039	2.425

〔3. 3〕

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
+0.02	−0.617 6	−0.018 19	−0.232 1	−0.881 6	2.906	0.240 2	2.535	2.932
+0.04	−0.559 5	−0.019 91	−0.237 5	−0.831 3	2.897	0.263 3	2.493	2.942
+0.06	−0.501 7	−0.021 78	−0.243 0	−0.781 9	2.887	0.288 5	2.448	2.954
+0.08	−0.444 0	−0.023 82	−0.249 0	−0.733 4	2.876	0.316 1	2.399	2.967
+0.10	−0.386 6	−0.026 04	−0.255 6	−0.685 9	2.864	0.346 3	2.347	2.981
+0.12	−0.329 5	−0.028 47	−0.262 9	−0.639 5	2.851	0.379 3	2.290	2.997
+0.14	−0.272 6	−0.031 11	−0.270 8	−0.594 3	2.837	0.415 5	2.230	3.014
+0.16	−0.216 0	−0.033 99	−0.279 5	−0.550 4	2.821	0.455 0	2.165	3.033
+0.18	−0.159 7	−0.037 12	−0.289 0	−0.507 8	2.804	0.498 2	2.095	3.052
+0.20	−0.103 8	−0.040 52	−0.299 5	−0.466 6	2.786	0.545 4	2.021	3.074
+0.22	−0.048 3	−0.044 22	−0.310 9	−0.426 9	2.765	0.596 7	1.942	3.099
+0.24	+0.006 7	−0.048 24	−0.323 4	−0.388 9	2.742	0.653 1	1.859	3.124
+0.26	+0.061 3	−0.052 59	−0.337 0	−0.352 6	2.718	0.714 4	1.771	3.153
+0.28	+0.115 4	−0.057 31	−0.352 0	−0.318 1	2.691	0.781 2	1.679	3.184
+0.30	+0.169 0	−0.062 44	−0.368 3	−0.285 5	2.662	0.853 8	1.582	3.218
+0.32	+0.221 9	−0.067 93	−0.386 2	−0.254 8	2.630	0.932 8	1.482	3.255
+0.34	+0.274 1	−0.073 89	−0.405 7	−0.226 2	2.595	1.019	1.378	3.296
+0.36	+0.325 7	−0.080 31	−0.427 0	−0.199 7	2.557	1.112	1.272	3.337
+0.38	+0.376 4	−0.087 22	−0.450 2	−0.175 3	2.516	1.212	1.165	3.383
+0.40	+0.426 3	−0.094 65	−0.475 5	−0.153 1	2.471	1.321	1.056	3.431
+0.42	+0.475 2	−0.102 62	−0.503 1	−0.133 1	2.423	1.438	0.947	3.483
+0.44	+0.523 2	−0.111 17	−0.533 1	−0.115 2	2.371	1.564	0.840	3.537
+0.46	+0.570 0	−0.120 32	−0.565 7	−0.099 5	2.315	1.700	0.736	3.594
+0.48	+0.615 7	−0.130 10	−0.601 2	−0.085 8	2.255	1.844	0.636	3.651
+0.50	+0.660 2	−0.140 54	−0.639 6	−0.073 9	2.192	1.997	0.540	3.709
+0.52	+0.703 4	−0.151 67	−0.681 2	−0.064 0	2.124	2.163	0.452	3.767
+0.54	+0.745 2	−0.163 52	−0.726 1	−0.055 8	2.052	2.337	0.371	3.819
+0.56	+0.785 4	−0.176 16	−0.774 7	−0.049 1	1.978	2.521	0.299	3.866
+0.58	+0.824 2	−0.189 63	−0.827 0	−0.043 8	1.898	2.715	0.237	3.904
+0.60	+0.861 4	−0.204 01	−0.883 1	−0.039 6	1.817	2.920	0.181	3.927
+0.62	+0.896 8	−0.219 42	−0.943 9	−0.036 5	1.732	3.135	0.136	3.925
+0.63	+0.913 9	−0.227 45	−0.975 8	−0.035 3	1.689	3.246	0.116	3.919
+0.64	+0.930 6	−0.235 90	−1.008 8	−0.034 2	1.646	3.361	0.098	3.898
+0.65	+0.946 8	−0.244 72	−1.043 0	−0.033 3	1.603	3.480	0.083	3.863
+0.66	+0.962 5	−0.253 94	−1.078 4	−0.032 5	1.560	3.601	0.069	3.812

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
+0.67	+0.977 9	−0.263 61	−1.115 1	−0.031 9	1.516	3.728	0.057	3.741
+0.68	+0.992 8	−0.273 89	−1.152 9	−0.031 1	1.173	8.861	0.047	3.651
+0.69	+1.007 3	−0.281 83	−1.192 2	−0.031 0	1.130	4.001	0.038	3.532
+0.695	+1.014 4	−0.290 61	−1.212 5	−0.030 8	1.408	4.074	0.034	3.460
+0.700	+1.021 1	−0.296 62	−1.233 0	−0.030 6	1.387	4.150	0.030	3.378
+0.705	+1.028 3	−0.302 90	−1.251 0	−0.030 5	1.366	4.228	0.027	3.285
+0.710	+1.035 1	−0.309 51	−1.275 3	−0.030 4	1.346	4.311	0.024	3.180
+0.715	+1.041 8	−0.316 49	−1.297 1	−0.030 2	1.326	4.398	0.021	3.061
+0.720	+1.048 4	−0.323 91	−1.319 3	−0.030 1	1.307	4.490	0.018	2.926
+0.725	+1.054 8	−0.331 87	−1.342 0	−0.030 1	1.288	4.589	0.016	2.773
+0.730	+1.061 [illegible]	−0.340 51	−1.365 2	−0.030 0	1.271	4.696	0.013	2.600
+0.735	+1.067 6	−0.350 01	−1.389 0	−0.029 9	1.251	4.815	0.011	2.401

(3.41)

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
+0.20	−0.101 0	−0.010 17	−0.299 5	−0.466 7	2.786	0.545 2	2.022	3.078
+0.22	−0.048 5	−0.014 17	−0.310 9	−0.427 0	2.765	0.596 6	1.943	3.102
+0.24	+0.006 7	−0.018 17	−0.323 4	−0.389 0	2.743	0.652 8	1.860	3.129
+0.26	+0.061 3	−0.052 52	−0.337 1	−0.352 7	2.717	0.711 2	1.772	3.158
+0.28	+0.115 1	−0.057 23	−0.352 0	−0.318 1	2.690	0.780 9	1.680	3.189
+0.30	+0.168 9	−0.062 32	−0.368 1	−0.285 5	2.661	0.853 6	1.583	3.224
+0.32	+0.221 8	−0.067 83	−0.386 2	−0.254 8	2.629	0.932 5	1.483	3.261
+0.34	+0.274 1	−0.073 77	−0.405 7	−0.226 2	2.594	1.018	1.380	3.301
+0.36	+0.325 6	−0.080 17	−0.427 0	−0.199 6	2.556	1.111	1.273	3.344
+0.38	+0.376 3	−0.087 07	−0.450 2	−0.175 3	2.515	1.212	1.166	3.391
+0.40	+0.426 2	−0.094 47	−0.475 5	−0.153 1	2.471	1.320	1.057	3.441
+0.42	+0.475 1	−0.102 42	−0.503 1	−0.133 0	2.422	1.437	0.949	3.494
+0.44	+0.523 0	−0.110 94	−0.533 1	−0.115 1	2.370	1.563	0.842	3.550
+0.46	+0.569 9	−0.120 01	−0.565 7	−0.099 3	2.315	1.698	0.738	3.606
+0.48	+0.615 6	−0.129 77	−0.601 0	−0.085 6	2.255	1.842	0.637	3.669
+0.50	+0.660 1	−0.140 11	−0.639 4	−0.073 8	2.191	1.996	0.543	3.731
+0.52	+0.703 2	−0.151 18	−0.680 9	−0.063 8	2.124	2.160	0.454	3.793
+0.54	+0.745 0	−0.162 92	−0.725 9	−0.055 5	2.052	2.333	0.373	3.851
+0.56	+0.785 3	−0.175 11	−0.774 3	−0.048 8	1.976	2.516	0.301	3.911
+0.58	+0.821 0	−0.188 67	−0.826 5	−0.043 4	1.898	2.708	0.238	3.962
+0.60	+0.861 2	−0.202 77	−0.882 7	−0.039 3	1.815	2.910	0.183	4.002

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
+0.62	+0.896 6	−0.217 79	−0.943 0	−0.036 1	1.731	3.122	0.138	4.026
+0.64	+0.930 4	−0.233 84	−1.007 6	−0.033 7	1.641	3.311	0.101	4.026
+0.65	+0.946 6	−0.242 30	−1.041 7	−0.032 8	1.600	3.459	0.085	4.013
+0.66	+0.962 4	−0.251 09	−1.076 8	−0.032 0	1.556	3.577	0.071	3.989
+0.67	+0.977 7	−0.260 21	−1.113 2	−0.031 3	1.511	3.698	0.060	3.953
+0.68	+0.992 6	−0.269 82	−1.150 8	−0.030 8	1.467	3.823	0.049	3.900
+0.69	+1.007 1	−0.279 88	−1.189 7	−0.030 3	1.423	3.953	0.040	3.828
+0.70	+1.021 1	−0.290 51	−1.229 9	−0.030 0	1.379	4.089	0.032	3.731
+0.71	+1.034 6	−0.301 84	−1.271 5	−0.029 7	1.336	4.231	0.026	3.604
+0.715	+1.041 3	−0.307 83	−1.292 8	−0.029 6	1.314	4.306	0.023	3.527
+0.720	+1.047 8	−0.314 06	−1.314 6	−0.029 5	1.293	4.384	0.020	3.440
+0.725	+1.054 2	−0.320 59	−1.336 7	−0.029 4	1.273	4.465	0.018	3.341
+0.730	+1.060 5	−0.327 44	−1.359 2	−0.029 3	1.252	4.550	0.015	3.229
+0.735	+1.066 7	−0.334 70	−1.382 2	−0.029 2	1.232	4.639	0.013	3.102
+0.740	+1.072 8	−0.342 43	−1.405 6	−0.029 2	1.213	4.734	0.012	2.958
+0.745	+1.078 9	−0.350 75	−1.429 6	−0.029 1	1.195	4.837	0.010	2.794
+0.750	+1.084 8	−0.359 81	−1.454 0	−0.029 1	1.177	4.950	0.008	2.609
+0.755	+1.090 6	−0.369 81	−1.479 1	−0.029 0	1.162	5.075	0.007	2.397

[3. 5]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
+0.20	−0.101 2	−0.040 40	−0.299 7	−0.466 9	2.786	0.545 1	2.023	3.085
+0.22	−0.048 7	−0.044 08	−0.311 1	−0.427 2	2.765	0.596 2	1.944	3.107
+0.24	+0.006 4	−0.048 08	−0.323 5	−0.389 1	2.743	0.652 4	1.861	3.134
+0.26	+0.061 0	−0.052 42	−0.337 2	−0.352 8	2.718	0.713 5	1.773	3.164
+0.28	+0.115 1	−0.057 11	−0.352 1	−0.318 2	2.691	0.780 2	1.681	3.196
+0.30	+0.168 6	−0.062 19	−0.368 4	−0.285 6	2.662	0.852 7	1.583	3.233
+0.32	+0.221 5	−0.067 67	−0.386 3	−0.254 9	2.630	0.931 6	1.485	3.269
+0.34	+0.273 8	−0.073 59	−0.405 7	−0.226 2	2.595	1.017	1.382	3.310
+0.36	+0.325 3	−0.079 97	−0.427 0	−0.199 6	2.557	1.110	1.276	3.355
+0.38	+0.376 0	−0.086 84	−0.450 2	−0.175 2	2.516	1.210	1.168	3.402
+0.40	+0.425 9	−0.094 21	−0.475 5	−0.152 9	2.471	1.319	1.060	3.454
+0.42	+0.471 8	−0.102 11	−0.503 0	−0.132 8	2.423	1.435	0.951	3.509
+0.44	+0.522 8	−0.110 58	−0.532 9	−0.114 8	2.371	1.561	0.845	3.568
+0.46	+0.569 6	−0.119 62	−0.565 5	−0.099 0	2.315	1.695	0.741	3.629
+0.48	+0.615 3	−0.129 27	−0.600 8	−0.085 2	2.255	1.839	0.641	3.694

[illegible]	[illegible]	[illegible]	(4)	(5)	(6)	(7)	(8)	(9)
				[3. 6]				
+0.08	−0.444 7	−0.023 68	−0.249 5	−0.731 2	2.876	[illegible]	[illegible]	2.981
+0.10	−0.387 3	−0.025 90	−0.256 1	−0.686 7	[illegible]	[illegible]	[illegible]	[illegible]
+0.12	−0.330 1	−0.028 31	−0.263 1	−0.640 3	[illegible]	[illegible]	2.292	[illegible]
+0.14	[illegible]	−0.030 93	[illegible]	−0.595 0	2.837	[illegible]	[illegible]	3.029
+0.16	−0.216 7	−0.033 79	−0.280 0	[illegible]	2.822	0.454 1	2.167	3.048
+0.18	−0.160 4	−0.036 90	[illegible]	−0.508 3	[illegible]	0.497 [illegible]	2.098	[illegible]
+0.20	[illegible]	−0.040 28	−0.299 9	[illegible]	2.786	0.544 [illegible]	[illegible]	3.091
+0.22	−0.049 0	−0.043 94	−0.311 2	−0.427 1	2.765	0.595 6	[illegible]	[illegible]
+0.24	+0.006 2	−0.047 93	−0.323 7	−0.389 3	[illegible]	[illegible]	[illegible]	3.144
+0.26	+0.060 8	−0.052 24	−0.337 3	[illegible]	[illegible]	0.712 8	1.776	3.174
+0.28	+0.114 9	[illegible]	−0.352 2	[illegible]	2.690	0.779 4	1.683	3.208
+0.30	+0.168 4	−0.061 97	−0.368 5	−0.285 6	2.661	0.851 8	1.587	3.243
+0.32	+0.221 3	−0.067 43	−0.386 1	[illegible]	2.629	0.930 5	1.487	3.282
[illegible]	+0.273 5	−0.073 32	[illegible]	−0.226 1	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	+0.325 0	−0.079 66	[illegible]	[illegible]	[illegible]	1.108	1.279	[illegible]
+0.38	+0.375 8	−0.086 48	[illegible]	−0.174 9	[illegible]	1.208	[illegible]	3.422
+0.40	+0.425 6	−0.093 79	[illegible]	−0.152 6	[illegible]	1.317	[illegible]	3.476
+0.42	[illegible]	−0.101 63	−0.502 9	−0.132 1	2.422	1.433	[illegible]	3.534
+0.44	[illegible]	−0.110 02	−0.532 8	−0.114 1	[illegible]	[illegible]	0.849	3.598
[illegible]	[illegible]	−0.118 96	[illegible]	[illegible]	2.314	1.692	[illegible]	3.666
+0.48	[illegible]	−0.128 48	−0.600 6	−0.084 6	[illegible]	[illegible]	0.645	[illegible]
				[3. 7]				
[illegible]	−2.280 0	−0.001 39	[illegible]	[illegible]	2.993	0.018 26	[illegible]	[illegible]
−0.52	−2.220 1	−0.001 53	[illegible]	−2.408 7	2.992	0.020 02	[illegible]	[illegible]
−0.50	[illegible]	−0.001 68	−0.186 1	[illegible]	2.991	0.021 95	[illegible]	[illegible]
[illegible]	−2.100 5	−0.001 84	[illegible]	−2.290 5	2.991	0.024 06	[illegible]	[illegible]
−0.46	−2.040 7	−0.002 01	[illegible]	[illegible]	[illegible]	0.026 38	[illegible]	[illegible]
−0.44	−1.980 9	−0.002 21	−0.187 6	−2.172 6	2.989	[illegible]	[illegible]	[illegible]
−0.42	[illegible]	[illegible]	[illegible]	−2.113 8	2.988	0.031 72	2.936	[illegible]
−0.40	−1.861 4	[illegible]	−0.188 9	[illegible]	2.987	0.034 77	2.929	[illegible]

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
−0.38	−1.801 7	−0.002 91	−0.189 6	−1.996 7	2.985	0.038 13	2.923	2.859
−0.36	−1.742 0	−0.003 19	−0.190 4	−1.938 3	2.981	0.041 80	2.915	2.860
−0.34	−1.682 3	−0.003 49	−0.191 3	−1.880 1	2.982	0.045 83	2.907	2.862
−0.32	−1.622 7	−0.003 83	−0.192 3	−1.822 0	2.981	0.050 25	2.899	2.861
−0.30	−1.563 1	−0.004 20	−0.193 3	−1.764 1	2.979	0.055 09	2.889	2.867
−0.28	−1.503 6	−0.004 60	−0.194 5	−1.706 5	2.977	0.060 40	2.878	2.869
−0.26	−1.444 1	−0.005 04	−0.195 7	−1.649 0	2.974	0.066 22	2.867	2.872
−0.24	−1.384 6	−0.005 52	−0.197 1	−1.591 8	2.972	0.072 60	2.854	2.875
−0.22	−1.325 2	−0.006 05	−0.198 0	−1.534 9	2.969	0.079 59	2.841	2.879
−0.20	−1.265 8	−0.006 63	−0.200 3	−1.478 2	2.966	0.087 26	2.825	2.882
−0.18	−1.206 6	−0.007 26	−0.202 1	−1.421 9	2.963	0.095 66	2.809	2.887
−0.16	−1.147 3	−0.007 96	−0.204 1	−1.365 9	2.959	0.104 9	2.791	2.891
−0.14	−1.088 2	−0.008 72	−0.206 3	−1.310 2	2.955	0.115 0	2.771	2.896
−0.12	−1.029 1	−0.009 55	−0.208 7	−1.255 0	2.951	0.126 0	2.750	2.901
−0.10	−0.970 2	−0.010 46	−0.211 4	−1.200 2	2.916	0.138 1	2.727	2.907
−0.08	−0.911 3	−0.011 45	−0.214 3	−1.146 0	2.941	0.151 4	2.702	2.914
−0.06	−0.852 5	−0.012 51	−0.217 5	−1.092 2	2.935	0.166 0	2.674	2.921
−0.04	−0.793 9	−0.013 72	−0.220 9	−1.039 0	2.929	0.181 9	2.644	2.929
−0.02	−0.735 4	−0.015 02	−0.224 7	−0.986 5	2.922	0.199 4	2.611	2.938
0.00	−0.677 0	−0.016 44	−0.228 9	−0.934 6	2.915	0.218 5	2.576	2.947
+0.02	−0.618 8	−0.017 98	−0.233 5	−0.883 4	2.906	0.239 4	2.538	2.957
+0.04	−0.560 8	−0.019 69	−0.238 5	−0.833 1	2.897	0.262 4	2.496	2.969
+0.06	−0.503 0	−0.021 53	−0.244 0	−0.783 6	2.887	0.287 5	2.451	2.981
+0.08	−0.445 3	−0.023 54	−0.250 0	−0.735 0	2.876	0.314 9	2.404	2.995
+0.10	−0.387 9	−0.025 74	−0.256 6	−0.687 5	2.864	0.345 0	2.351	3.010

PAPER CHROMATOGRAPHY OF INORGANIC IONS BY USING ORGANIC ANALYTICAL REAGENTS. VIII.

Paper Chromatography of Cations with Dithizone. (Part 2)

Hideo NAGAI

(Received Oct. 31, 1959)

A number of metal dithizonates have been separated chromatographically by using the hydrophobic solvents, such as chloroform or carbon tetrachloride, as the developing solvents. In these procedures the metal dithizonates were developed in a completely dissolved form in the developing solvents. On the other hand, the hydrophilic solvents, such as water or alcohols, had comparatively minor dissolving power to the dithizonates, so they were inferior or almost unsuitable developing solvents for the usual chromatography of dithizonates. The author intended to utilize these minor dissolving power rather positively, as in the analogous chromatographic system of precipitation chromatography, which was studied in the separation of the metal oxinates. "Low solubility chromatography" may be the proper name for this special kind of chromatography. In the case of the weakly acidified metal nitrates solution given for test, 0.1N nitric acid and aceton (10:1) showed the best result for the developer. Cu^{2+}, Cd^{2+}, Hg^{2+}, Pb^{2+}, and Bi^{3+} were separated almost clearly.

Introduction

When dithizone was used as the color developing reagent for metals, it was usual to choose the hydrophobic solvents such as chloroform or carbontetrachloride for dissolving solvent. Accordingly these hydrophobic solvents have hitherto been used widely as the developing solvent for the chromatographic procedure to separate dithizonates. And in this procedure alumina was most preferably used as the adsorber of the chromatography[1]. Intending to bring this technique in the field of paper chromatography, the author tried to use the filter paper modified with alumina as the adsorber, but the result was inferier to that of the original column method[2].

On the other hand, the author developed the precipitation chromatography to separate metal oxinates[3)~8)]. In this procedure the dissolving power of the developing solvent was intentionally deducted. Therefore, the precipitation chromatography or analogous procedure might be able to be utilized rather positively in the case of the separation of the minor soluble substances in the solvent. On this point of view, the author constructed the dithizone-hydrophilic solvent system, consulting the procedure of the precipitation chromatography of the metal oxinates. The result was not inferor to that of alumina column-hydrophobic solvent system and far easier in operation. In this paper, the separation of Group 2 (Hg^{2+}, Pb^{2+}, Bi^{3+}, Cu^{2+}, and Cd^{2+}) was described.

Experimental

The experiment was done as an analogous procedure as precipitation chromatography; the test (sample) solution applied in a spot on a filter paper impregnated with the complexing agent was developed with an adequate developing solvent.

1) Impregnation of the filter paper with dithizone. Sheets of filter paper were dipped in the organic solvent with dithizone dissolved in it, and then dried in the open air. The filter paper, thus treated, seemed to retain some amount of solvent in it, because the filter paper colored in green; the color is the organic solvent containing dithizone in it. Moreover, the filter paper treated with hydrophobic solvent showed still the hydrophobic property during the fairly long time, inspite of its apparent dryness. For this reason, it would be better to adopt the hydrophilic solvents for the treatment of the filter paper in the following experiment in which aqueous solution was used as the developing solvent of the chromatography. Methanol, ethanol, and acetone were tried for the solvent in the treatment of the filter paper, and acetone was found to be most preferable, probably because of its largest dissolving power to the dithizone in these three organic solvents.

The treatment was as follows. Dithizone was weighed ca. 0.5 g., and dissolved in 100 ml. of acetone. One hundred sheets of circular filter paper, 9 cm. in diameter (Toyo filter paper No. 5B), were dipped in the acetone with dithizone dissolved in it. After a while, the paper sheets were taken out, dried in the open air, and left overnight there. Filter paper, thus treated, was stored in a closed vessel.

2) Preparation of the test (sample) solution. Available cations consisted of cupric nitrate, cadmium nitrate, lead nitrate, bismuth nitrate and mercuric chloride. Each of the metal salts was dissolved in about 100 ml. of water so as to contain ca. 10 g. cation in it. Bismuth alone was dissolved in 6N nitric acid instead of pure water. When these metal salt solutions were used for the experiment, they were diluted so as to contain ca. 1 mg. in 1 ml. of each test solution, and weakly acidified with nitric acid (about 0.06N nitric acid solution).

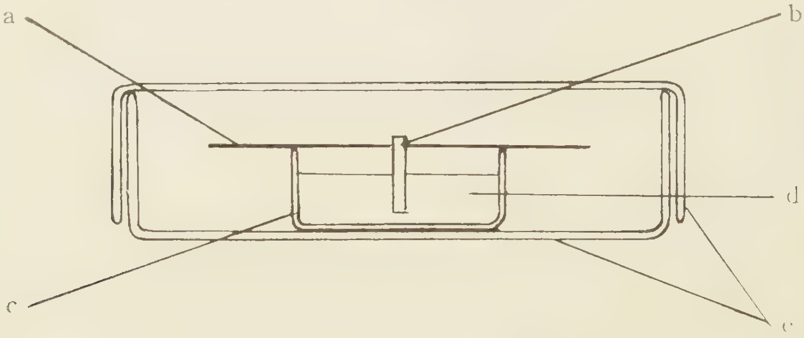

Fig. 1 a: Filter paper impregnated with dithizone.
b: Strip of untreated filter paper.
c: Small Petri dish.
d: Developing solvent.
e: Large Petri dish.

3) Application of sample solution and development. The sample solution was spotted at the center of the filter paper treated with dithizone. A slit (1∼1.5 mm. in length) was cut with a blade in the middle of the spot, and inserted a narrow strip of untreated filter paper in the slit so as to bring the developing solvent to the dithizone treated filter paper. Then the developing of chromatography was done as illustrated in fig. 1. As the developing solvent penetrated into the filter paper, the dithizone contained in the filter paper dissolved little by little into the developing solvent. It resembled the dropwise addition of the complexing agent to the test solution in the complexometric titration. Accordingly, the most stable dithizonate in the test solution appeared at first, then followed the next stable complex salt, and so on, forming the concentric circular bands of the dithizonates.

4) Developing solvent.

Water: The test solution containing three components–Hg^{2+}, Cu^{2+}, and one of the following three cations, Cd^{2+}, Pb^{2+}, and Bi^{3+}–could be identified. However, the test solution containing four or more components could not be identified on account of the confusion of the chromatogram, except inner two bands corresponding to Hg^{2+} and Cu^{2+}. The figure of the chromatogram was not concentric circular but irregularly waving at the periphery. Following the analogous deduction of the precipitation chromatography, it is probably owing to the deficient dissolving power of the developing solvent (water) to the metal dithizonates. From this point of view, the study to find the adequate developing solvent in this chromatographic system was directed towards the more powerful dissolving solvents to the dithizonates.

Water-n-Butanol and Water-iso-Butanol: Water saturated with n- or iso-butanol was a good developing solvent for the precipitation chromatography with oxine, as reported before[3], but in this study they had no suffucient dissolving power to the dithizonates, though they showed somewhat better results than water for the developing solvent.

Water–Acetone: The irregular waving of the periphery of the chromatogram, probably caused by the deficient dissolving power of the developing solvent, disappeared when acetone was added to the water in ca. 1/50 in volume and used for the developing solvent. At first, the increase of the proportion of the acetone in developing solvent caused the improvement of the separation of the bands of the chromatogram.

When acetone was increased to the ratio, water: acetone=20:1, the separation of the three components, described above, was attained more clearly than in the case of water as the developer. And moreover, the identification of Hg^{2+}, Cu^{2+}, Cd^{2+}, and Pb^{2+} came to be possible.

When the developing solvent, water: aceton=10:1, was used, the separability of the bands was improved more but no other new combination of cations could be separated.

Water: acetone=5:1 was used as the developing solvent, the separability of the bands was almost the same as the case described above (water: aceton=10:1), but a faint redish brown ring appeared near the developing front. This ring was thought to be the perfectly dissolved dithizonate in the developing solvent. Accordingly, the

chromatography of usual type also ought to be considered in the further increase of acetone, in addition to the special kind of chromatography described in this paper; "low solubility chromatography" might be the proper name for this special kind of chromatography. However, in this constitution of the developing solvent, most of the cations to be analysed were still retained in the region of the inner low solubility chromatogram.

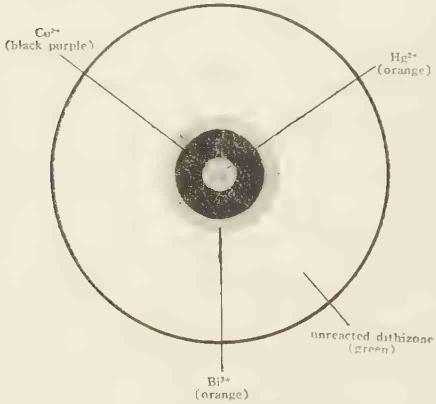

Fig. 2 A

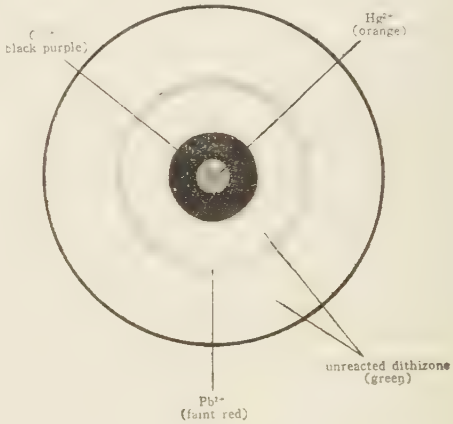

Fig. 2 B

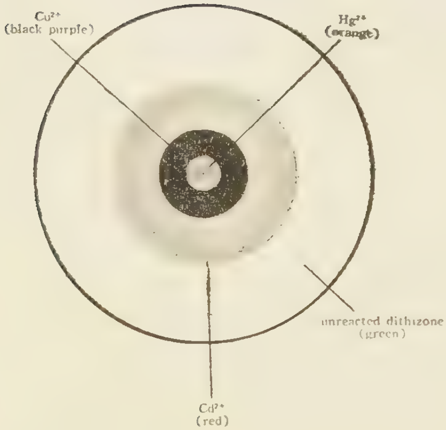

Fig. 2 C

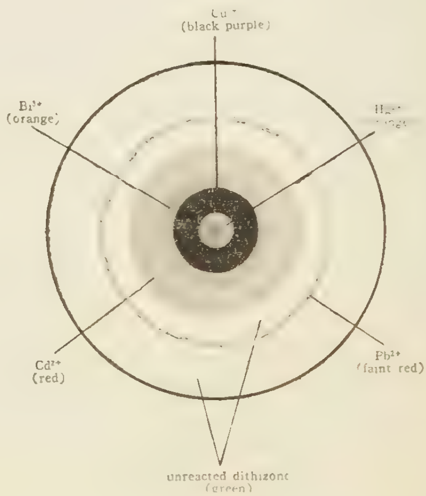

Fig. 3

Water: acetone=1:1 was used as the developer, the new combination of cations, Hg^{2+}, Cu^{2+}, Bi^{3+}, and Pb^{2+} was also identified.

Water: acetone=1:2 was used, the low solubility chromatogram was formed early in the development, and then its bands, except inner two (Hg and Cu) decolored

and the faint outer ring became deeper in color and larger in area. When the development was continued further, Hg band, situated inside the Cu band, was also dissolved, and came to mix in the Cu band. During the whole period of the development, the Cu band was retained almost unchanged.

Water: acetone=1:7 was used, all the metal dithizonates were almost perfectly dissolved in the developing solvent, so the chromatogram formed was of the type of hitherto studied, and its separability of the bands was far inferior to that of low solubility chromatography.

From the result of the experiments described above, the following combination of four cations Hg^{2+}, Cu^{2+}, Bi^{3+}, and Cd^{2+} could not be successfully identified.

Water–Acetone–Nitric Acid. Following the analogy of the precipitation chromatography, the low solubility chromatography has also the necessity to control the *p*H as well as the dissolving power of the developing solvent[1]. Considering the result of the acetone-water developing solvent system, it must be better to use the near constitution of water: acetone=10:1, so various concentration of aqueous nitric acid solutions, adding 1/10 acetone in their volume, tried for the developing solvent.

When developed with 0.05N nitric acid: acetone=10:1, the test solutions used in the previous experiments and found to be able to identified, were all separated more effectively, and, moreover, the identification of Hg–Cu–Bi–Cd and Hg–Cu–Bi–Cd–Pb were attained, but the separation between Bi and its neighbouring cations was not satisfactory.

0.1N nitric acid: acetone=10:1 was used as the developing solvent, the separations of cations, described above, were all able to be achieved almost completely. The separations of three cations, Hg–Cu–Bi, Hg–Cu–Pb, and Hg–Cu–Cd were illustrated in fig. 2A, B, and C. And the separation of five cations was attained following from the prospect of the overlapping of these three figures (fig. 3).

Using the nitric acid 0.2N in concentration, the separation between Bi and its neighbouring cations became more clearly, but the coloration of the Pb band became insufficient.

Discussion

The author intended to discuss the mechanism of the low solubility chromatography described in this paper comparing with the precipitation chromatography, reported before. In the case of the precipitation chromatography, the most predominant factor for the separability of cations was considered to be the difference of the stabilities of the complex salts[6]. When the stable complex salt was formed, it was taken out from the reaction system forming precipitation, acceralating the reaction to form the same precipitation. Finally, the chromatogram, thus formed, appeared just to exaggerate the differences among the complex salts. This phenomenon was very effective to separate cations.

On the other hand, in the case of the low solubility chromatography, the reaction between the complexing agent and the cations appeared to be performed principally in the region of the developing front. And the complex salt formed in this region,

slowed down its developing speed on account of the low solubility of the salt in the developing solvent. Finally the salt retained backward from the developing front. The complex salt, thus formed, might be said to be "taken out" from the reaction system, though its form was not the precipitation, as in the case of the precipitation chromatography. Accordingly, the analogous exaggeration of the differences of the stabilities among the complex salts was expected as in the case of the precipitation chromatography. Moreover, the difference of the solubilities of the salts was also expected for the separation factor as the case of the usual soluble type chromatography.

Essentially, the low solubility chromatography might be said to search the optimum condition to separate cations adequately utilizing the properties of the two different kinds of chromatography-soluble type chromatography and precipitational type chromatography.

The temperature of the developing solvents used in this experiment was 25°～30°C.

The contents of this paper have mainly been published in Nippon Kagaku Zasshi, Vol. 80, p. 617 (1959).

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1) T. Ashizawa, Repts. Balneol. Lab. Okayama Univ. No. 5, 1 (1951).
2) H. Nagai, Kumamoto J. of Science Series A, Vol. 2, No. 1, 100 (1954).
3) H. Nagai, Kumamoto J. of Science Series A, Vol. 2, No. 3, 304 (1955).
4) H. Nagai, Kumamoto J. of Science Series A, Vol. 3, No. 1, 81 (1957).
5) H. Nagai, Kumamoto J. of Science Series A, Vol. 3, No. 2, 167 (1957).
6) H. Nagai, Kumamoto J. of Science Series A, Vol. 3, No. 2, 171 (1957).
7) H. Nagai, Kumamoto J. of Science Series A, Vol. 3, No. 2, 176 (1957).
8) H. Nagai, Kumamoto J. of Science Series A, Vol. 3, No. 2, 181 (1957).

ERRATA

In the papers that appeared in these Journals, following corrections should be given.

RESEARCH ON THE COMPENSATED PENDULUM. Ryuzo ADACHI Vol. 1, No.2 (1953)

page	line		Should be read
27	17	$I_t\ h_t$	I_t/h_t

ON A PROOF OF FUNDAMENTAL FORMULA CONCERNING REFRACTION METHOD. Ryuzo ADACHI Vol. 2, No. 1 (1954)

page	line		Should be read
19	8	$\cos B_1$	$\cos \beta_1$
22	8	$\cdots = \sin a_{n-1}$	$\cdots \sin a_{n-1}$

ON THE NUMERICAL SOLUTION OF THE SIMULTANEOUS DIFFERENTIAL EQUATIONS UNDER SOME CONDITIONS. Ryuzo ADACHI Vol. 2, No. 1 (1954)

page	line		Should be read
39	16	which is	which were

APPROXIMATE FORMULAS FOR DEFINITE INTEGRALS AND DIFFERENTIAL COEFFICIENTS. Ryuzo ADACHI Vol. 2, No. 2 (1955)

page	line		Should be read
196	20	$\int_a^{a+ih} f(x)dx$	$\int_a^{a+jh} f(x)dx$
196	21	$\sum_{j=0}^{p}$	$\sum_{i=0}^{p}$
197	8	$\lvert R_{p\cdot i}\rvert$	$\lvert R_{p\cdot j}\rvert$
205	14	$\sum_{i=1}^{p} ib_{i\cdot j}=0$	$\sum_{i=1}^{p} ib_{i\cdot j}=1$
207	9	$\sum_{i=1}^{p} i^2c_{i\cdot j}=0$	$\sum_{i=1}^{p} i^2c_{i\cdot j}=2$
208	14, 18	place	places
208	24	$f(x)_{(p+1)}\equiv 0$	$f^{(p+1)}(x)\equiv 0$

ON THE FORM OF A SURFACE OF LIQUID WHICH IS IN EQUILIBRIUM UNDER PRESSURE AND SURFACE-TENSION. Ryuzo ADACHI Vol. 2, No. 2 (1955)

page	line		Should be read
211	14	$(CZ+C')^2$	$(CZ^2+C')^2$
211	20	ect.	etc.

A METHOD ON THE NUMERICAL SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS.
Ryuzo ADACHI Vol. 2, No. 3 (1955)

page	line		Should be read
245	22	becouse	because
249	8	empoy	employ

A PROBLEM ON THE SEISMIC PROSPECTING. Ryuzo ADACHI Vol. 2, No. 4 (1956)

page	line		Should be read
366	7	$-\{\beta Fx_0^2+(\gamma^2+\beta E)x_0]$	$-\{\beta Fx_0^2+(\gamma^2+\beta E)x_0\}]$

A METHOD OF EXPLORATION ON THE SEISMIC PROSPECTING. Ryuzo ADACHI Vol. 3, No. 1 (1957)

page	line		Should be read
1	23	$\frac{v_2\sqrt{1+y'(x_0)^2}}{v_1[v_1-Vy'(x_0)]}$	$\frac{v_2\sqrt{1+y'(x_0)^2}}{v_1[v_1-Vy'(x_0)]}x_0$
1	25	$\frac{v_2\sqrt{1+y'(x)^2}}{v_1[V-v_1y'(x)]}$	$\frac{v_2\sqrt{1+y'(x)^2}}{v_1[V-v_1y'(x)]}y(x)$
2	25	exit angle	emergent angle
9	11	$(\beta x+C'')$	$(\beta X+C'')$
10	2	$\cdots\cdots=\varphi(X)+\cdots\cdots$	$\cdots\cdots=\Psi(X)+\cdots\cdots$
11	15	numeriaal	numerical
12	13	$-\frac{1}{k}-[\gamma^2(l-n)+(\alpha-\beta)\{C+\beta(l-m)]\geqq\bar{a}$	$\frac{1}{k}[\gamma^2(l-n)+(\alpha-\beta)\{C+\beta(l-m)\}]\geqq\bar{a}$
22	18	$t=\varphi(X)$	$\bar{t}=\bar{\varphi}(\bar{X})$
23	3	$t=\varphi(X)$	$\bar{t}=\bar{\varphi}(\bar{X})$

FUNDAMENTAL RELATION ON THE SEISMIC PROSPECTING AND A METHOD OF EXPLORATION. Ryuzo ADACHI Vol. 3, No. 1 (1957)

page	line		Should be read
25	1	NO	ON
25	11	de	be
25	12	odserved	observed
25	14	bn	be

ON THE MAGNIFICATION OF THE RECORD OF A VIBRATION BY AN ELECTROMAGNETIC TYPE TRANSDUCER AND A GALVANOMETER. Ryuzo ADACHI Vol. 4, No. 1 (1959)

page	line		Should be read
20	6	Fig. 3 and	Fig. 3, A and

ON THE WAVE MOTION PROPAGATED ALONG AN ELASTIC CYLINDER WITH INFINITE LENGTH. Ryuzo ADACHI Vol. 4, No. 2 (1959)

page	line		Should be read
137	8	$\sqrt{1-\frac{b_j}{k}}$	$\sqrt{1-\frac{b_j^2}{k^2}}$
139	2	$\frac{2i_\kappa p}{h_2} J_1(ap)$	$\frac{2i_\kappa p}{h^2} J_1(ap)$

ERRATA

QUASI-STATIONARY PROCESS (I). ON ENTROPY PRODUCTION By Shigeich FUJITA
Kumamoto J. Science A. Vol. 2, No. 1, (1954) 51

Page	Place	Wrong	Correct
52	line 1	ate	are
52	line 12	whern	when
52	line 17	$\frac{T_1-T_2}{2}$	$\frac{T_1+T_2}{2}$
53	line 12	………(7)	(to be eliminated)
53	line 16	………(8)	(to be eliminated)

QUASI-STATIONARY PROCESS (III). CHANGES OF STATE OF GAS BY THE SLOW CHANGES OF ITS BOUNDARY CONDITIONS By Shigeichi FUJITA
Kumamoto J. Science A. Vol. 2, No. 3, (1955) 277

Page	Place	Wrong	Correct
285	line 4	$v_0{}^{\frac{c}{\gamma}t}$	$v_0 e^{\frac{c}{\gamma}t}$

ASCENSION OF A SMALL MASS OF AIR AND THE CHANGES IN ITS STATE (II) WET AIR By Shigeichi FUJITA
Kumamoto J. Science A. Vol. 2, No. 4, (1957) 32

Page	Place	Wrong	Correct
36	line 14	p	p_1
36	line 17	p	p_1
36	line 20	vapour	water
37	line 6 from bottom	table	Table
38	line 2	(16)	(20)
42	line 7	hieght	height

ON OSIMA'S BLOCKS OF CHARACTERS OF GROUPS OF FINITE ORDER

Kenzo IIZUKA

(Received June 30, 1960)

Let $\mathfrak{G}$ be a group of finite order g and p be a fixed rational prime. M. OSIMA, in his paper [5], introduced a concept of blocks of group characters with regard to a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ ("$\mathfrak{H}$ blocks"). Let $\mathfrak{H}_0$ be the maximal normal subgroup of $\mathfrak{G}$ contained in $\mathfrak{H}$. It is well known that the irreducible characters[1] ϕ_1, ϕ_2, $\cdots\cdots$, ϕ_k of $\mathfrak{H}_0$ are distributed into the classes $\mathfrak{B}_1, \mathfrak{B}_2, \cdots\cdots, \mathfrak{B}_s$ of associated characters in $\mathfrak{G}$. If $\mathfrak{B}'_1, \mathfrak{B}'_2, \cdots\cdots, \mathfrak{B}'_{s'}$ are the classes of associated irreducible characters of $\mathfrak{H}_0$ in $\mathfrak{H}$, then each class $\mathfrak{B}_\sigma$ is a collection of classes $\mathfrak{B}'_\rho$. Let $\chi_1, \chi_2, \cdots\cdots, \chi_n$ be the irreducible characters of $\mathfrak{G}$ and θ_1, θ_2, $\cdots\cdots$, θ_h be those of $\mathfrak{H}$. As is well known, there corresponds to each character χ_i exactly one class $\mathfrak{B}_\sigma$ such that

$$\chi_i(H_0)=s_{i\sigma}\sum_{\phi_\mu\subset\mathfrak{B}_\sigma}\phi_\mu(H_0) \qquad (H_0\in\mathfrak{H}_0),$$

where $s_{i\sigma}$ is a positive rational integer. If a class $\mathfrak{B}_\sigma$ corresponds to a character χ_i in this sense, we say that χ_i belongs to $\mathfrak{B}_\sigma$ by counting χ_i in $\mathfrak{B}_\sigma$. We also say that θ_λ belongs to $\mathfrak{B}_\sigma$ if θ_λ belongs to $\mathfrak{B}'_\rho$ contained in $\mathfrak{B}_\sigma$. We set

$$\chi_i(H)=\sum_{\lambda}^{h} r_{i\lambda}\theta_\lambda(H) \qquad (H\in\mathfrak{H}),$$

where the $r_{i\lambda}$ are non-negative rational integers. As is easily seen, if $r_{i\lambda}\neq 0$, then χ_i and θ_λ belong to the same class $\mathfrak{B}_\sigma$. Hence, χ_i and χ_j belong to the same class $\mathfrak{B}_\sigma$ if and only if χ_i and χ_j are connected by a chain $\chi_i, \chi_r, \cdots\cdots, \chi_t, \chi_j$ such that any two consecutive $\chi_l(H)$ and $\chi_m(H)$ of the chain have an irreducible constituent θ_λ in common, i. e. $r_{l\lambda}\neq 0$ and $r_{m\lambda}\neq 0$. Thus the classes $\mathfrak{B}_\sigma$ are the $\mathfrak{H}$-blocks of $\mathfrak{G}$ in OSIMA's sense.[2] From the definition of the classes $\mathfrak{B}_\sigma$, we have the following:

LEMMA 1. *Two characters χ_i and χ_j belong to the same $\mathfrak{H}$-block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ if and only if*

$$\frac{\chi_i(H_0)}{\chi_i(1)}=\frac{\chi_j(H_0)}{\chi_j(1)}$$

for all elements H_0 of $\mathfrak{H}_0$, where 1 *denotes the identity of the group* $\mathfrak{G}$. ([5])

Henceforth the term "block of a group" will always mean block with regard to a p-Sylow subgroup of the group. While BRAUER's blocks with regard to a rational prime q will be referred as q-blocks.

The purpose of this paper is to consider a connection between the blocks of $\mathfrak{G}$ and

1) The term "irreducible character" will always mean absolutely irreducible ordinary character.
2) Cf. [5].

those of the normalizer $\mathfrak{N}(R)$ of a p-regular element R in $\mathfrak{G}$.[3]

NOTATION: $\mathfrak{G}$ denotes a group of finite order g and p is a fixed rational prime. Ω is the field of g-th roots of unity. $K_1, K_2, \cdots\cdots, K_n$ are the classes of conjugate elements in $\mathfrak{G}$; there are n distinct irreducible characters $\chi_1, \chi_2, \cdots\cdots, \chi_n$ of $\mathfrak{G}$. $\mathfrak{N}(G)$ denotes the normalizer of an element G in $\mathfrak{G}$; the order of $\mathfrak{N}(G)$ is denoted by $n(G)$. For a rational prime q, any element G of $\mathfrak{G}$ is written uniquely as $G=SQ=QS$, where S is a q-regular element and Q is an element whose order is a power of q; S is called the q-regular factor of G and Q is called the q-factor of G.

1. Let $\mathfrak{P}$ be a p-Sylow subgroup of $\mathfrak{G}$ and $\mathfrak{P}_0$ be the maximal normal p-subgroup of $\mathfrak{G}$. We denote by $\mathfrak{B}_1, \mathfrak{B}_2, \cdots\cdots, \mathfrak{B}_s$ the blocks of $\mathfrak{G}$ with regard to $\mathfrak{P}$. For each block $\mathfrak{B}_\sigma$, we set

$$\Delta_\sigma = \sum_{\chi_i \in \mathfrak{B}_\sigma} e_i, \tag{1.1}$$

where e_i denotes the primitive idempotent of the center Z of the group ring of $\mathfrak{G}$ over Ω which belongs to χ_i, $i=1,2,\cdots\cdots,n$. Let $G_1, G_2, \cdots\cdots, G_n$ be a complete system of representatives for the classes $K_1, K_2, \cdots\cdots, K_n$. If we interprete each class K_ν as the sum of all its elements, then we may write

$$\Delta_\sigma = \sum_\nu a_\nu^\sigma K_\nu, \tag{1.2}$$

where

$$a_\nu^\sigma = \frac{1}{g} \sum_{\chi_i \in \mathfrak{B}_\sigma} \chi_i(1)\bar{\chi}_i(G_\nu).^{4)} \tag{1.3}$$

We denote by Φ_σ the sum of all irreducible characters ϕ_μ of $\mathfrak{P}_0$ which belong to $\mathfrak{B}_\sigma$ and denote by ϕ^* the character of $\mathfrak{G}$ induced by a character ϕ of $\mathfrak{P}_0$. If we set

$$\chi_i(P_0) = s_{i\sigma}\Phi_\sigma(P_0) \qquad (P_0 \in \mathfrak{P}_0) \tag{1.4}$$

where $s_{i\sigma}$ is a positive rational integer, then, by Frobenius' theorem on induced characters, we have

$$\phi_\mu^*(G) = \sum_{\chi_i \in \mathfrak{B}_\sigma} s_{i\sigma}\chi_i(G) \qquad (G \in \mathfrak{G}) \tag{1.5}$$

for each irreducible character ϕ_μ of $\mathfrak{P}_0$ belonging to $\mathfrak{B}_\sigma$.

LEMMA 2. 1) $a_\nu^\sigma = 0$ *for all classes* K_ν *which are not contained in* $\mathfrak{P}_0$. 2) *All* $(\mathfrak{P}_0:1)a_\nu^\sigma$ *are algebraic integers.*

PROOF. 1) By the above formulae (1.3)—(1.5), we have

$$a_\nu^\sigma = \frac{1}{g}\Phi_\sigma(1)\bar{\phi}_\mu^*(G_\nu),$$

where $\phi_\mu \in \mathfrak{B}_\sigma$. Since each class K_ν containing an element of $\mathfrak{P}_0$ is contained in $\mathfrak{P}_0$, $\phi_\mu^*(G_\lambda)=0$ for all $K_\lambda \not\subseteq \mathfrak{P}_0$. Hence we have $a_\nu^\sigma = 0$ for these classes K_λ.

3) A summary of the results obtained herein will appear in [4].

4) If α is a complex number, the conjugate complex number of α is denoted by $\bar{\alpha}$.

2) For $K_\nu \subseteqq \mathfrak{P}_0$, we have

$$a_\nu^\sigma = \frac{1}{g} \bar{\Phi}_\sigma(G_\nu) \phi_\mu^*(1) = \frac{1}{(\mathfrak{P}_0:1)} \bar{\Phi}_\sigma(G_\nu) \phi_\mu(1),$$

where $\phi_\mu \in \mathfrak{B}_\sigma$. Since $\Phi_\sigma(G_\nu)$ and $\phi_\mu(1)$ are algebraic integers, it follows from this formula and 1) in this lemma that all $(\mathfrak{P}_0:1)a_\nu^\sigma$ are algebraic integers, if $K_\nu \subseteqq \mathfrak{P}_0$ or not.

THE CONVERSE OF LEMMA 2. *If, for a set $\mathfrak{B}$ of characters χ_i, the idempotent $\varDelta = \sum_{\chi_i \in \mathfrak{B}} e_i$ of Z is expressed as a linear combination of classes K_ν contained in $\mathfrak{P}_0$, then $\mathfrak{B}$ is a collection of blocks $\mathfrak{B}_\sigma$ of $\mathfrak{G}$.*

PROOF. Suppose $\mathfrak{B} \cap \mathfrak{B}_\sigma$ is not vacuous and $\mathfrak{B} \not\supseteq \mathfrak{B}_\sigma$. Then we may select two characters χ_i and χ_j of $\mathfrak{B}_\sigma$ such that $\chi_i \in \mathfrak{B}$ and $\chi_j \notin \mathfrak{B}$. For these characters, we have $\omega_i(\varDelta)=1$ and $\omega_j(\varDelta)=0$, where ω_i and ω_j are the linear characters of Z which belong to e_i and e_j, respectively. On the other hand, we have $\omega_i(\varDelta)=\omega_j(\varDelta)$, because we have $\omega_i(K_\nu)=\omega_j(K_\nu)$ for all $K_\nu \subseteqq \mathfrak{P}_0$ by Lemma 1. Therefore $\mathfrak{B}$ must be a collection of blocks $\mathfrak{B}_\sigma$ of $\mathfrak{G}$.

2. Let q be an arbitrarily fixed rational prime, different from p, and $\mathfrak{q}$ be a prime ideal in Ω dividing q. For each q-block B_τ of $\mathfrak{G}$, we consider the primitive idempotent η_τ of the center Z_0 of the group ring of $\mathfrak{G}$ over the ring $\mathfrak{o}_\mathfrak{q}$ of $\mathfrak{q}$-integers: $\eta_\tau = \sum_{\chi_i \in B_\tau} e_i$. If we set

$$\eta_\tau = \sum_\nu b_\nu^\tau K_\nu \tag{2.1}$$

then, as is well known, b_ν^τ vanishes for all q-singular classes K_ν of $\mathfrak{G}$ and all the coefficients b_ν^τ are $\mathfrak{q}$-integers. The converse also holds in the following form: If, for a set B of characters χ_i, the idempotent $\eta = \sum_{\chi_i \in B} e_i$ is expressed as a linear combination of the classes K_ν of $\mathfrak{G}$ with $\mathfrak{q}$-integral coefficients, then B is a collection of q-blocks B_τ of $\mathfrak{G}$. Therefore, it follows from Lemma 2 that *each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ is a collection of q-blocks B_τ of $\mathfrak{G}$.*

Let Q be an arbitrarily given element of $\mathfrak{G}$ whose order is a power of q. Let $B^{(\tau)}(Q)$ be the collection of q-blocks B_ρ of $\mathfrak{N}(Q)$, which determine a q-block B_τ of $\mathfrak{G}$ in BRAUER's sense, and $\hat{\eta}_\rho$ be the primitive idempotent of the center $\hat{Z}_0$ of the group ring of $\mathfrak{N}(Q)$ over the ring $\mathfrak{o}_\mathfrak{q}$ of $\mathfrak{q}$-integers which is associated with a q-block B_σ of $\mathfrak{N}(Q)$. We set $\hat{\eta}^{(\tau)} = \sum_{\hat{B}_\rho \subseteq B^{(\tau)}(Q)} \hat{\eta}_\rho$ and $\eta_\tau^0 = \sum_\nu b_\nu^\tau K_\nu^0$, where K_ν^0 is the sum of all elements in $K_\nu \cap \mathfrak{N}(Q)$, $\nu=1,2,\cdots\cdots,n$. It is well known that

$$\eta_\tau^0 \equiv \hat{\eta}^{(\tau)} \pmod{\mathfrak{q}Z_0}.^{5)} \tag{2.2}$$

If we set $\mathfrak{B}^{(\sigma)}(Q) = \bigcup_{B_\tau \subseteq \mathfrak{B}_\sigma} B^{(\tau)}(Q)$, then we have the following:

LEMMA 3. *Each $\mathfrak{B}^{(\sigma)}(Q)$ is a collection of blocks $\mathfrak{B}_\gamma$ of $\mathfrak{N}(Q)$.*

5) Cf. (4.16) in [2] and [8, p. 181].

PROOF. Suppose $\hat{\mathfrak{B}}_\gamma \frown \mathfrak{B}^{(\sigma)}(Q)$ is not vacuous and $\hat{\mathfrak{B}}_\gamma \not\equiv \mathfrak{B}^{(\sigma)}(Q)$. Then we may choose two irreducible characters $\hat{\chi}_i$ and $\hat{\chi}_j$ of $\mathfrak{N}(Q)$ in $\hat{\mathfrak{B}}_\gamma$ such that $\hat{\chi}_i \in \mathfrak{B}^{(\sigma)}(Q)$ and $\hat{\chi}_j \notin \mathfrak{B}^{(\sigma)}(Q)$. If we set $\Delta'_\sigma = \sum_{B_\tau \subseteq \mathfrak{B}_\sigma} \eta'_\tau$, then we have $\Delta'_\sigma = \sum_\nu a^\sigma_\nu K'_\nu$, where the coefficients a^σ_ν are given by (1.2) for the block $\mathfrak{B}_\sigma$. By Lemma 2, Δ^0_σ is a linear combination of classes $\hat{K}_\mu$ of $\mathfrak{N}(Q)$ with $\mathfrak{q}$-integral coefficients which are contained in $\mathfrak{P}_0 \frown \mathfrak{N}(Q)$. Since $\mathfrak{P}_0 \frown \mathfrak{N}(Q)$ is contained in the maximal normal p-subgroup $\hat{\mathfrak{P}}_0$ of $\mathfrak{N}(Q)$, we have $\hat{\omega}_i(\Delta^0_\sigma) = \hat{\omega}_j(\Delta^0_\sigma)$ by Lemma 1, where $\hat{\omega}_i$ and $\hat{\omega}_j$ are the linear characters of the center Z of the group ring of $\mathfrak{N}(Q)$ over Ω which belong to $\hat{\chi}_i$ and $\hat{\chi}_j$, respectively. Setting $\hat{\Delta}^{(\sigma)} = \sum_{B_\tau \subseteq \mathfrak{B}_\sigma} \hat{\eta}^{(\tau)}$, by (2.2) we have

$$\hat{\omega}_i(\hat{\Delta}^{(\sigma)}) \equiv \hat{\omega}_j(\hat{\Delta}^{(\sigma)}) \pmod{\mathfrak{q}}. \tag{2.3}$$

On the other hand, we have $\hat{\omega}_i(\hat{\Delta}^{(\sigma)}) = 1$ and $\hat{\omega}_j(\hat{\Delta}^{(\sigma)}) = 0$, hence

$$\hat{\omega}_i(\hat{\Delta}^{(\sigma)}) \not\equiv \hat{\omega}_j(\hat{\Delta}^{(\sigma)}) \pmod{\mathfrak{q}}. \tag{2.4}$$

(2.4) contradicts with (2.3), therefore $\mathfrak{B}^{(\sigma)}(Q)$ is a collection of blocks of $\mathfrak{N}(Q)$. This completes the proof.

3. Let R be a p-regular element of $\mathfrak{G}$. If the order of R is a product of powers of r distinct rational primes $q_1, q_2, \cdots\cdots, q_r$, then R is uniquely decomposed into

$$R = Q_1 Q_2 \cdots\cdots Q_r \qquad (Q_i Q_j = Q_j Q_i), \tag{3.1}$$

where Q_i is the q_i-factor of R, $i = 1, 2, \cdots\cdots, r$. Let a block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ be given arbitrarily. First, applying Lemma 3 for Q_1, $\mathfrak{B}_\sigma$ and $\mathfrak{G}$, we have a collection $\mathfrak{B}^{(\sigma)}(Q_1)$ of blocks of $\mathfrak{G}^{(1)} = \mathfrak{N}(Q_1)$. Secondly, working similarly for Q_2, $\mathfrak{B}^{(\sigma)}(Q_1)$, and $\mathfrak{G}^{(1)}$, we have a collection $\mathfrak{B}^{(\sigma)}(Q_1, Q_2)$ of blocks of $\mathfrak{G}^{(2)} = \mathfrak{N}(Q_1 Q_2)$. Continuing this process, we have finally a collection $\tilde{\mathfrak{B}}^{(\sigma)} = \mathfrak{B}^{(\sigma)}(Q_1, Q_2, \cdots\cdots\cdots, Q_r)$ of blocks $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}} = \mathfrak{N}(R)$. If a block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$ belongs to the collection $\tilde{\mathfrak{B}}^{(\sigma)}$, we say that *the block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ is determined by hte block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$*. It follows immediately from Theorem 1 that $\tilde{\mathfrak{B}}^{(\sigma)}$ is independent of the order of $Q_1, Q_2, \cdots\cdots, Q_r$.

Let $S(R)$ be the p-regular section of R in $\mathfrak{G}$ ("Oberklasse")[6], i. e. the set of all elements of $\mathfrak{G}$ whose p-regular factors are conjugate to R in $\mathfrak{G}$. Let $\tilde{K}_1, \tilde{K}_2, \cdots\cdots, \tilde{K}_{\tilde{n}}$ be the classes of conjugate elements in $\tilde{\mathfrak{G}}$; we may assume that the p-regular section $\tilde{S}(1)$ of 1 in $\tilde{\mathfrak{G}}$ is the union of the first v classes $\tilde{K}_\alpha$. We may choose a complete system of representatives $P_1, P_2, \cdots\cdots, P_v$ for the classes $\tilde{K}_1, \tilde{K}_2, \cdots\cdots, \tilde{K}_v$ in a given p-Sylow subgroup $\tilde{\mathfrak{P}}$ of $\tilde{\mathfrak{G}}$. As is easily seen, any two elements RP_α and RP_β with $\alpha \neq \beta$ can not belong to the same class K_ν of $\mathfrak{G}$. Hence, arranging the classes K_ν of $\mathfrak{G}$ in a suitable order, we may assume that RP_α belongs to K_α, $\alpha = 1, 2, \cdots\cdots, v$; $S(R)$ is the union of $K_1, K_2, \cdots\cdots, K_v$. Further we may assume that the maximal normal p-subgroup $\tilde{\mathfrak{P}}_0$ of $\tilde{\mathfrak{G}}$ is the union of $\tilde{K}_1, \tilde{K}_2, \cdots\cdots, \tilde{K}_u$, $1 \leqq u \leqq v$. We denote by $S_0(R)$ the union of

6) Cf. [11].

$K_1, K_2, \cdots\cdots, K_u$ and denote by $S_1(R)$ the union of $K_{u+1}, K_{u+2}, \cdots\cdots, K_v$; $S(R)=S_0(R) \cup S_1(R)$.

There are u distinct blocks $\tilde{\mathfrak{B}}_1, \tilde{\mathfrak{B}}_2, \cdots\cdots, \tilde{\mathfrak{B}}_u$ of $\tilde{\mathfrak{G}}$ with regard to $\tilde{\mathfrak{P}}$ ([5]), as is immediately seen from Lemma 1. For each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$, let J_σ be given by (1.1). Similarly, for each block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$, we may define an idempotent $\tilde{J}_\rho$ of the center $\tilde{Z}$ of the group ring of $\tilde{\mathfrak{G}}$ over Ω. We set $\tilde{J}^{(\sigma)}=\sum_{\tilde{\mathfrak{B}}_\rho \subseteq \mathfrak{B}^{(\sigma)}} \tilde{J}_\rho$ and set

$$K_\mu J_\sigma=\sum_{\nu=1}^{n} a_{\mu\nu}^{\sigma} K_\nu \qquad (\mu=1,2,\cdots\cdots,n). \tag{3.2}$$

We then have the following:

THEOREM 1. *For $\alpha=1,2,\cdots\cdots,u$, we have*

$$K_\alpha J_\sigma=\sum_{\beta=1}^{u} a_{\alpha\beta}^{\sigma} K_\beta \tag{3.3}$$

and

$$\tilde{K}_\alpha \tilde{J}^{(\sigma)}=\sum_{\beta=1}^{u} a_{\alpha\beta}^{\sigma} \tilde{K}_\beta. \tag{3.4}$$

For $\alpha=u+1, u+2, \cdots\cdots, v$, we have

$$K_\alpha J_\sigma=\sum_{\beta=u+1}^{v} a_{\alpha\beta}^{\sigma} K_\beta \tag{3.3'}$$

and

$$\tilde{K}_\alpha \tilde{J}^{(\sigma)}=\sum_{\beta=u+1}^{v} a_{\alpha\beta}^{\sigma} \tilde{K}_\beta. \tag{3.4'}$$

PROOF. According to Lemma 2, we may set

$$\tilde{K}_\alpha \tilde{J}^{(\sigma)}=\sum_{\beta} \tilde{a}_{\alpha\beta}^{(\sigma)} \tilde{K}_\beta \qquad (\alpha=1,2,\cdots\cdots,v), \tag{3.5}$$

where β ranges over $1,2,\cdots\cdots,u$ for $\alpha=1,2,\cdots\cdots,u$ and ranges over $u+1, u+2,\cdots\cdots, v$ for $\alpha=u+1, u+2, \cdots\cdots, v$. Denote by $J^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_f)$ the idempotent of the center of the group ring of $\mathfrak{G}^{(f)}$ over Ω which is associated with $\mathfrak{B}^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_f)$, $f=1,2,\cdots\cdots, r$; $\tilde{J}^{(\sigma)}=J^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_r)$. Since any two elements $Q_r P_\alpha$ and $Q_r P_\beta$ with $\alpha \neq \beta$ can not belong to the same class of conjugate elements in $\mathfrak{G}^{(r-1)}$, if we denote by $K_\alpha^{(r-1)}$ the class of conjugate elements in $\mathfrak{G}^{(r-1)}$ which contains $Q_r P_\alpha$, $\alpha=1,2,\cdots\cdots, v$, then $K_1^{(r-1)}, K_2^{(r-1)}, \cdots\cdots, K_v^{(r-1)}$ are v distinct classes of $\mathfrak{G}^{(r-1)}$. First, considering $\mathfrak{B}^{(\sigma)}$ and $\mathfrak{B}^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_{r-1})$ as collections of q_r-blocks of $\tilde{\mathfrak{G}}$ and $\mathfrak{G}^{(r-1)}$, respectively, by Theorem 2 in [3] we have

$$K_\alpha^{(r-1)} J^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_{r-1})=\sum_{\beta} \tilde{a}_{\alpha\beta}^{(\sigma)} K_\beta^{(r-1)}$$

with the same proviso as (3.5). Secondly, denoting by $K_\alpha^{(r-2)}$ the classes of conjugate elements in $\mathfrak{G}^{(r-2)}$ which contains $Q_{r-1} Q_r P_\alpha$, $\alpha=1,2,\cdots\cdots, v$, we have

$$K_{\alpha}^{(r-2)} J^{(\sigma)}(Q_1, Q_2, \cdots\cdots, Q_{r-2}) = \sum_{\beta} \tilde{a}_{\alpha\beta}^{(\sigma)} K_{\beta}^{(r-2)}$$

with the same proviso as above. Continuing this process, we have finally (3.3) and (3.3′) with $a_{\alpha\beta}^{\sigma} = \tilde{a}_{\alpha\beta}^{(\sigma)}$ $(\alpha, \beta = 1, 2, \cdots\cdots, v)$. This completes the proof.

4. In this section, we shall use the notations in the preceding sections. The elements $P_1, P_2, \cdots\cdots, P_v$ need not form a complete system of representatives for the classes of conjugate elements in $\mathfrak{P}$. However, we may construct such a system by adding further elements P to the set $P_1, P_2, \cdots\cdots, P_v$. Each P is conjugate to a certain P_α in $\tilde{\mathfrak{G}}$, where α is uniquely determined, $\alpha = 1, 2, \cdots\cdots, v$. We denote by the elements P belonging to P_α by $P_\alpha = P_\alpha^{(0)}, P_\alpha^{(1)}, P_\alpha^{(2)}, \cdots\cdots, P_\alpha^{(l_\alpha)}, l_\alpha \geqq 0$. Let $\tilde{\chi}_1, \tilde{\chi}_2, \cdots\cdots, \tilde{\chi}_{\tilde{n}}$ be the irreducible characters of $\tilde{\mathfrak{G}}$ and $\tilde{\theta}_1, \tilde{\theta}_2, \cdots\cdots, \tilde{\theta}_{\tilde{h}}$ be those of $\tilde{\mathfrak{P}}$. We set

$$\tilde{\chi}_j(P) = \sum_{\lambda=1}^{\tilde{h}} \tilde{r}_{j\lambda} \tilde{\theta}_\lambda(P) \qquad (P \in \tilde{\mathfrak{P}}) \tag{4.1}$$

and, after M. OSIMA [6], [7], we define the constants $r_{i\lambda}^{R}$ by

$$\chi_i(RP) = \sum_{\lambda=1}^{\tilde{h}} r_{i\lambda}^{R} \tilde{\theta}_\lambda(P) \qquad (P \in \tilde{\mathfrak{P}}). \tag{4.2}$$

For $\alpha, \beta = 1, 2, \cdots\cdots, v$, since we see from (3.1) that

$$a_{\alpha\beta}^{\sigma} = \frac{1}{n(RP_\alpha)} \sum_{\chi_i \in \mathfrak{B}_\sigma} \chi_i(RP_\alpha) \bar{\chi}_i(RP_\beta) = \frac{1}{n(RP_\alpha)} \sum_{\chi_i \in \mathfrak{B}_\sigma} \chi_i(RP_\alpha^{(\gamma)}) \bar{\chi}_i(RP_\beta^{(\delta)}),$$

$a_{\alpha\beta}^{\sigma}$ is expressed as

$$a_{\alpha\beta}^{\sigma} = \frac{1}{n(RP_\alpha)} \sum_{\lambda,\mu=1}^{h} \tilde{\theta}_\lambda(P_\alpha^{(\gamma)}) \bar{\tilde{\theta}}_\mu(P_\beta^{(\delta)}) \sum_{\chi_i \in \mathfrak{B}_\sigma} r_{i\lambda}^{R} \bar{r}_{i\mu}^{R}. \tag{4.3}$$

On the other hand, from Theorem 1 we see, for $\alpha, \beta = 1, 2, \cdots\cdots, v$,

$$\begin{aligned} a_{\alpha\beta}^{\sigma} &= \frac{1}{n(RP_\alpha)} \sum_{\tilde{\chi}_j \in \tilde{\mathfrak{B}}^{(\sigma)}} \tilde{\chi}_j(P_\alpha) \bar{\tilde{\chi}}_j(P_\beta) \\ &= \frac{1}{n(RP_\alpha)} \sum_{\tilde{\theta}_\lambda, \tilde{\theta}_\mu \in \tilde{\mathfrak{B}}^{(\sigma)}} \tilde{\theta}_\lambda(P_\alpha^{(\gamma)}) \bar{\tilde{\theta}}_\mu(P_\beta^{(\delta)}) \sum_{\tilde{\chi}_j \in \tilde{\mathfrak{B}}^{(\sigma)}} \tilde{r}_{j\lambda} \tilde{r}_{j\mu} \\ &= \frac{1}{n(RP_\alpha)} \sum_{\tilde{\theta}_\lambda, \tilde{\theta}_\mu \in \tilde{\mathfrak{B}}^{(\sigma)}} \tilde{\theta}_\lambda(P_\alpha^{(\gamma)}) \bar{\tilde{\theta}}_\mu(P_\beta^{(\delta)}) \sum_{j=1}^{\tilde{n}} \tilde{r}_{j\lambda} \tilde{r}_{j\mu}. \end{aligned}$$

Setting

$$\tilde{w}_{\lambda\mu} = \sum_{j=1}^{\tilde{n}} \tilde{r}_{j\lambda} \tilde{r}_{j\mu} \qquad (\lambda, \mu = 1, 2, \cdots\cdots, \tilde{h}),$$

we have

$$a_{\alpha\beta}^{\sigma} = \frac{1}{n(RP_\alpha)} \sum_{\tilde{\theta}_\lambda, \tilde{\theta}_\mu \in \tilde{\mathfrak{B}}^{(\sigma)}} \tilde{\theta}_\lambda(P_\alpha^{(\gamma)}) \bar{\tilde{\theta}}_\mu(P_\beta^{(\delta)}) \tilde{w}_{\lambda\mu}. \tag{4.4}$$

Since (4.3) and (4.4) hold for any pair of γ and δ ($\gamma=0,1,2,\cdots\cdots,l_\alpha$; $\delta=0,1,2,\cdots,l_\beta$), we have

$$\sum_{\chi_i\in\mathfrak{B}_\sigma} r^R_{i\lambda}\, r^R_{i\mu}=\begin{cases}\tilde{w}_{\lambda\mu} & (\tilde{\theta}_\lambda,\ \tilde{\theta}_\mu\in\tilde{\mathfrak{B}}_\rho\subseteqq\tilde{\mathfrak{B}}^{(\sigma)}),\\ 0 & (\text{elsewhere}).\end{cases}\tag{4.5}$$

In particular, we have

$$\sum_{\chi_i\in\mathfrak{B}_\sigma} r^R_{i\lambda}\, r^R_{i\lambda}=0 \qquad (\tilde{\theta}_\lambda\notin\tilde{\mathfrak{B}}^{(\sigma)}),$$

hence

$$r^R_{i\lambda}=0 \qquad (\chi_i\in\mathfrak{B}_\sigma,\ \tilde{\theta}_\lambda\notin\tilde{\mathfrak{B}}^{(\sigma)}).$$

Thus we obtain the following:

THEOREM 2. *If an irreducible character $\tilde{\theta}_\lambda$ of $\tilde{\mathfrak{P}}$ belongs to a block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$, then $r^R_{i\lambda}$ can be different from zoro only for irreducible characters χ_i of $\mathfrak{G}$ which belong to the block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$ determined by the block $\tilde{\mathfrak{B}}_\rho$ of $\tilde{\mathfrak{G}}$.*

It is easy to see that $a^\sigma_{\mu\nu}=0$ implies

$$\sum_{\chi_i\in\mathfrak{B}_\sigma}\chi_i(G_\mu)\bar{\chi}_i(G_\nu)=0$$

Hence, by Theorem 1, we have a refinement of some of the orthogonality relations for group characters.[7]

THEOREM 3. 1) *If two elements L and M of $\mathfrak{G}$ belong to different p-regular sections of $\mathfrak{G}$, then*

$$\sum_{\chi_i\in\mathfrak{B}_\sigma}\chi_i(L)\bar{\chi}_i(M)=0\tag{4.6}$$

for each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$.[8] ([10])

2) *If L and M belong to the same p-regular section $S(R)$ of $\mathfrak{G}$ and if exactly one of the p-factors of them belongs to the maximal normal p-subgroup of $\mathfrak{N}(R)$, then (4.6) also holds for each block $\mathfrak{B}_\sigma$ of $\mathfrak{G}$.*

From (3.3) and (3.3′), we see the following:

LEMMA 4. *If*

$$\sum_{\nu=1}^{n} a_\nu K_\nu J_\sigma=0,\tag{4.7}$$

where $a_\nu\in\Omega$, then

$$\sum_{K_\nu\in S(R)} a_\nu K_\nu J_\sigma=\sum_{K_\nu\in S(R)} a_\nu K_\nu J_\sigma=0$$

for each p-regular section $S(R)$ of $\mathfrak{G}$.

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7) Cf. [1]—[4], [9] and [10].

8) We have a refinement of this, which is a dual result of Theorem 2 in [1].

We shall describe this lemma in terms of the irreducible characters χ_i of $\mathfrak{G}$. (4.7) implies

$$\sum_{\nu=1}^{n} a_\nu \omega_i(K_\nu) = \omega_i\left(\sum_{\nu=1}^{n} a_\nu K_\nu J_\sigma\right) = 0 \qquad (\chi_i \in \mathfrak{B}_\sigma).$$

where $\omega_i(K_\nu) = g\chi_i(G_\nu)/n(G_\nu)\chi_i(1)$. Conversely, if

$$\sum_{\nu=1}^{n} a_\nu \omega_i(K_\nu) = 0$$

for all $\chi_i \in \mathfrak{B}_\sigma$, where the a_ν are numbers of Ω depending only on the classes K_ν of $\mathfrak{G}$, then

$$\omega_j\left(\sum_{\nu=1}^{n} a_\nu K_\nu J_\sigma\right) = \sum_{\nu=1}^{n} a_\nu \omega_j(K_\nu)\omega_j(J_\sigma) = 0$$

for all irreducible characters χ_j of $\mathfrak{G}$. Hence (4.7) holds for these a_ν. Therefore Lemma 4 is equivalent to the following:

LEMMA 4′. *If, for all χ_i of $\mathfrak{B}_\sigma$,*

$$\sum_{\nu=1}^{n} a_\nu \omega_i(K_\nu) = 0$$

where the a_ν are numbers of Ω depending only on the classes K_ν of $\mathfrak{G}$, then

$$\sum_{K_\nu \subseteq S_0(R)} a_\nu \omega_i(K_\nu) = \sum_{K_\nu \subseteq S_1(R)} a_\nu \omega_i(K_\nu) = 0$$

for these χ_i, where R is an arbitrary p-regular element of $\mathfrak{G}$.

Evidently this lemma is equivalent to the following:

LEMMA 4″. *If, for all χ_i of $\mathfrak{B}_\sigma$,*

$$\sum_{\nu=1}^{n} a_\nu \chi_i(G_\nu) = 0$$

where the a_ν are numbers of Ω depending only on the classes K_ν of $\mathfrak{G}$, then

$$\sum_{K_\nu \subseteq S_0(R)} a_\nu \chi_i(G_\nu) = \sum_{K_\nu \subseteq S_1(R)} a_\nu \chi_i(G_\nu) = 0$$

for these χ_i, where R is an arbitrary p-regular element of $\mathfrak{G}$.

In consideration of Lemma 4″, the orthogonality relations for group characters imply the following refinement of some of them.

THEOREM 4. *If χ_i and χ_j are two irreducible characters of $\mathfrak{G}$ which belong to different blocks $\mathfrak{B}_\sigma$ of $\mathfrak{G}$, then*

$$\sum_{G \in S_0(R)} \chi_i(G)\bar{\chi}_j(G) = \sum_{G \in S_1(R)} \chi_i(G)\bar{\chi}_j(G) = 0$$

for each p-regular section $S(R)$ of $\mathfrak{G}$.[9]

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9) This is an improvement of a result in [10]. Cf. the papers quoted in foot-note 7).

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ON THE SOLUTION OF THE SECOND ORDER LINEAR DIFFERENCE EQUATION

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§ 1. Introduction

Difference equations are often employed for some problems in physics, technology and genetics etc., and most of them are second order linear.

The first order linear difference equation has been solved *in general*, but so far as the author knows, the second order linear difference equation has not yet been solved *in general*, and only some special cases have been solved (e. g., the cases in which the coefficients are constants or linear forms of the independent variable).

We can put the general form of the second order linear difference equation to

$$y_{x+2}+\alpha(x)y_{x+1}+\beta(x)y_x=\gamma(x) \quad \cdots\cdots\cdots \quad (1)$$

or

$$\Delta^2 y_x+a(x)\Delta y_x+b(x)y_x=c(x) \quad \cdots\cdots\cdots \quad (2),$$

where $y_x=y(x)$, $\Delta y_x=y_{x+1}-y_x$, $\Delta^2 y_x=\Delta y_{x+1}-\Delta y_x$ etc.,
and $\alpha(x)$, $\beta(x)$, $\cdots\cdots$, $c(x)$ are known functions of the independent variable x. Ordinarily the form (1) is used. The equations (1) and (2) are transformable with each other, therefore taking the equation (2) as the general form, we discussed its solution.

§ 2. Transformation of (2)

In order to transform (2) into a form $\Delta^2 y_x=p(x)y_x+q(x)$,
we put to

$$y(x)=u(x)v(x) \quad \cdots\cdots\cdots \quad (3),$$

then we get

$$v_{x+2}\Delta^2 u_x+\{2\Delta v_{x+1}+a(x)v_{x+1}\}\Delta u_x+\{\Delta^2 v_x+a(x)\Delta v_x+b(x)v_x\}u_x=c(x) \cdots\cdots \quad (4).$$

Therefore by determining v_x from $2\Delta v_{x+1}+a(x)v_{x+1}=0$, that is

$$v_x=K exp\left[\int log\left\{1-\frac{1}{2}a(x-1)\right\}\Delta x\right] \cdots\cdots\cdots \quad (5),$$

(4) becomes

$$\Delta^2 u_x=p(x)u_x+q(x) \cdots\cdots\cdots \quad (6)$$

where

$$\left.\begin{aligned} p(x)&=1-\frac{v_x}{v_{x+1}}\{1-a(x)+b(x)\} \\ q(x)&=c(x)\,v_{x+1} \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (7).$$

Since this transformation has no condition it is always possible, hence we can take the equation (6) as the general form of the second order linear difference equation. Moreover we can put to $0\leqq x$ with no loss of generality.

Accordingly, in the following, we consider the solution of (6), where $p(x)$, $q(x)$ are known given functions and $0\leqq x$.

§ 3. Lemmas

In the first place, we denote several lemmas referring to the seeking of the solution.

Definition:

For any function $f(x)$ we define

$$\Delta f(x)=f(x+1)-f(x)\cdots\cdots\cdots\cdots \text{ difference}$$

$$\text{if } \Delta F(x)=f(x)\cdots \begin{cases} \sum f(x)\Delta x=F(x)\cdots\cdots\cdots \text{ indefinite summation} \\ \sum_a^b f(x)\Delta x=F(b)-F(a)\cdots\cdots \text{ definite summation.} \end{cases}$$

Then there are following properties:

(**a**) if $\Psi(x)=\sum_a^x f(x)\Delta x$, we have $\Delta\Psi(x)=f(x)$ $\quad\cdots\cdots\cdots\cdots$ (i)

where $a=$constant,

(**d**) if $b-a=n=$ *integer*>0,

$$\sum_a^b f(x)\Delta x=\sum_0^n f(x+a)\Delta x=f(b-1)+f(b-2)+\cdots\cdots+f(a+1)+f(a) \quad\cdots\cdots \text{(ii)}$$

(**c**) if $P(x)$ is a periodic function with its period 1,

$$\Delta\{P(x)f(x)\}=P(x)\Delta f(x)\cdots\cdots\cdots\cdots \text{(iii)}$$

$$\sum_a^b P(x)f(x)\Delta x=P(a)\sum_a^b f(x)\Delta x \quad\cdots\cdots\cdots\cdots \text{(iv)}$$

where $b-a=$*integer*,

(**d**) if $0\leqq x=$*integer*,

$$\left|\sum_0^x f(x)\Delta x\right|\leqq\sum_0^x |f(x)|\Delta x \quad\cdots\cdots\cdots\cdots \text{(v)}$$

$$\mathop{S}_{0}^{x} f(x)\Delta x \leqq \mathop{S}_{0}^{x} \varphi(x)\Delta x \cdots\cdots \text{ for } f(x) \leqq \varphi(x) \qquad \text{(vi)}$$

$$\mathop{S}_{0}^{x} \Big\{ f(x) \pm \varphi(x) \Big\} \Delta x = \mathop{S}_{0}^{x} f(x)\Delta x \pm \mathop{S}_{0}^{x} \varphi(x)\Delta x \qquad \text{(vii)}$$

$$\left.\begin{aligned} &\mathop{S}_{0}^{x} x^r \Delta x < \frac{1}{r+1}\, x^{r+1} \cdots\cdots \text{ for } r>0 \\ &\mathop{S}_{0}^{x} x^r \Delta x = x \cdots\cdots \text{ for } r=0 \end{aligned}\right\} \qquad \text{(viii)}$$

(e) we define $F(x, r)$ such that

$$F(x, 1)=x, \quad F(x, r)=\frac{1}{r!}\, x(x-1)(x-2)\cdots\cdots(x-r+1) \qquad \text{(ix)},$$
$$\text{for } \quad r \geqq 2$$

then we have

$$\Delta F(x, r)=F(x, r-1), \quad r \geqq 2 \qquad \text{(x)}$$

if $x=$ *positive integer*

$$\left.\begin{aligned} &F(x, r)={}_xC_r \cdots\cdots \text{ for } r \leqq x \\ &F(x, r)=0 \quad \cdots\cdots \text{ for } r \geqq x+1 \end{aligned}\right\} \qquad \text{(xi)}$$

$$\mathop{S}_{0}^{x} F(x, r)\Delta x=F(x, r+1), \quad r \geqq 1 \qquad \text{(xii)}.$$

The proof of these lemmas are very easy.

§ 4. Operator L

Definition:

Let $p(x), f(x)$ be bounded in the interval $0 \leqq x \leqq 2N$ ($N=$ *integer*) and we define an operator L such that

$$\left.\begin{aligned} &L^0[f(x)]=f(x), \quad L^1[f(x)]=L[f(x)]=\mathop{S}_{\varepsilon}^{x}\Big\{\mathop{S}_{\varepsilon}^{x} p(x)f(x)\Delta x\Big\}\Delta x \\ &L^r[f(x)]=L\Big[L^{r-1}[f(x)]\Big], \quad r \geqq 2 \end{aligned}\right\} \qquad (8)$$

where n denotes the integer part of x and ε the decimal part, therefore $x=n+\varepsilon$ and $0 \leqq n=$ *integer*, $0 \leqq \varepsilon < 1$.

Then we have following theorems.

(i)

$$L[f(x)]=\sum_0^n\left\{\sum_0^m p(t+\varepsilon)f(t+\varepsilon)\Delta t\right\}\Delta m \quad \cdots\cdots\cdots\cdots \quad (9)$$

(ii) Let

$$|p(x)|\leqq P,\ |f(x)|\leqq K \quad (P,\ K=consts.)\ \text{for}\ 0\leqq x\leqq 2N$$

then we get

$$|L^r[f(x)]|\leqq KP^rF(n,2r) \quad \cdots\cdots\cdots\cdots \quad (10).$$

Proof:
By lemmas (v), (vi) and (xii)

$$|L[f(x)]|\leqq\sum_0^n\left|\sum_0^m p(t+\varepsilon)f(t+\varepsilon)\Delta t\right|\Delta m\leqq KP\sum_0^n m\Delta m=KPF(n,2)$$

$$|L^2[f(x)]|\leqq|L[KPF(n,2)]|\leqq KP^2F(n,4)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$|L^r[f(x)]|\leqq|L[KP^{r-1}F(n,2\overline{r-1})]|\leqq KP^rF(n,2r).$$

(iii)

$$L^r[f(x)]=0 \quad \text{for}\ N+1\leqq r \quad \cdots\cdots\cdots\cdots \quad (11).$$

Proof:

Since $n\leqq x\leqq 2N<2r$, by lemma (xi) we have

$$|L^r[f(x)]|\leqq KP^rF(n,2r)=0 \quad \therefore\ L^r[f(x)]=0.$$

§ 5. The solution of (6)

We consider the solution of

$$\Delta u_x=p(x)u_x+q(x) \quad \cdots\cdots\cdots\cdots \quad (6)$$

under the conditions that

$$u_\varepsilon=b(\varepsilon),\ \Delta u_\varepsilon=c(\varepsilon) \quad \cdots\cdots\cdots\cdots \quad (12),$$

where $p(x), q(x)$ are given functions in the interval $0\leqq x\leqq 2N$ and $b(x), c(x)$ are given functions in the interval $0\leqq x\leqq 1$.

In the first place, in consequence of $x-\varepsilon=n=integer$, the summations of both sides of (6) from ε to x are uniquely determined and we get

$$\Delta u_x=c(\varepsilon)+\sum_\varepsilon^x p(x)u_x\Delta x+\sum_\varepsilon^x q(x)\Delta x,$$

similarly

$$u_x=c(\varepsilon)x+b(\varepsilon)-\varepsilon c(\varepsilon)+L[u_x]+\int_\varepsilon^x\left\{\int_\varepsilon^x q(x)\Delta x\right\}\Delta x.$$

Putting

$$\left.\begin{aligned}&Q(x)=\int_\varepsilon^x\left\{\int_\varepsilon^x q(x)\Delta x\right\}\Delta x,\ B(\varepsilon)=b(\varepsilon)-\varepsilon c(\varepsilon)\\&\varphi_0(x)=c(\varepsilon)x+B(\varepsilon)+Q(x)\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots\quad(13)$$

gives

$$u_x=\varphi_0(x)+L[u_x],\ 0\leqq x\leqq 2N+2\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\quad(14).$$

Here, $\varepsilon=x-n=(x+1)-(n+1)$, therefore $B(\varepsilon)$ and $C(\varepsilon)$ are periodic functions of x with their periods *1*. Accordingly we can introduce (6) and (12) from (14) by using lemma (iii). Therefore (14) is equivalent to (6) and (12), hence we consider the solution of (14) in the following.
Put

$$Y(x)=\varphi_0(x)+\sum_{r=1}^{N+1}L^r[\varphi_0(x)],\ 0\leqq x\leqq 2N+2\cdots\cdots\cdots\cdots\cdots\cdots\quad(15)$$

then by (11) we get

$$L^{N+2}[\varphi_0(x)]=0,$$

therefore

$$L[Y(x)]=L[\varphi_0(x)]+\sum_{r=2}^{N+2}L^r[\varphi_0(x)]=\sum_{r=1}^{N+1}L^r[\varphi_0(x)]=Y(x)-\varphi_0(x)$$

that is

$$Y(x)=\varphi_0(x)+L[Y(x)].$$

This indicates that $Y(x)$ defined by (15) is the solution of (14). Next, put

$$\left.\begin{aligned}&Y_1(x)=x+\sum_{r=1}^{N+1}L^r[x],\ \ Y_2(x)=1+\sum_{r=1}^{N+1}L^r[1]\\&Y_3(x)=Q(x)+\sum_{r=1}^{N+1}L^r[Q(x)],\ \ 0\leqq x\leqq 2N+2\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\quad(16)$$

then we have

$$Y(x)=c(\varepsilon)Y_1(x)+B(\varepsilon)Y_2(x)+Y_3(x)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\quad(17).$$

From (16) we can easily derive following equations

$$\left.\begin{aligned}&Y_1(x)=x+L[Y_1(x)],\ \ Y_2(x)=1+L[Y_2(x)]\\&Y_3(x)=Q(x)+L[Y_3(x)],\ \ 0\leqq x\leqq 2N+2\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots\quad(18)$$

and

$$\left.\begin{array}{l}\Delta^2 Y_1(x)=p(x)Y_1(x),\quad \Delta^2 Y_2(x)=p(x)Y_2(x)\\ \Delta^2 Y_3(x)=p(x)Y_3(x)+q(x),\quad 0\leq x\leq 2N\end{array}\right\} \qquad (19)$$

Therefore $Y_1(x), Y_2(x)$ *are two solutions of*

$$\Delta^2 u_x=p(x)u_x \qquad (20)$$

and $Y_3(x)$ *is a particular solution of* (6).

Next suppose that there exists a constant k such that $Y_1(x)=kY_2(x)$, then by (18), we get

$$kY_2(x)=Y_1(x)=x+L[Y_1(x)]=x+kL[Y_2(x)]=x-k+kY_2(x).$$

that is $x=k$, this is a contradiction, hence $Y_1(x)$ and $Y_2(x)$ are linearly independent with each other.

§ 6. Example

We give a simple example and some remarks.

Example: Solve the equation

$$\Delta^2 u_x=a^2u_x,\quad a=const\neq 0 \qquad (i)$$
$$0\leq x\leq 2N$$

under the conditions

$$u_\varepsilon=b(\varepsilon),\quad \Delta u_\varepsilon=c(\varepsilon) \qquad (ii).$$

Solution.

For this example, we have $p(x)=a^2$, $q(x)=0$, therefore

$$Q(x)=0$$
$$L[1]=a^2F(n,2),\ L^2[1]=a^4F(n,4),\cdots\cdots L^r[1]=a^{2r}F(n,2r)$$
$$L[x]=a^2\{F(n,3)+\varepsilon F(n,2)\},\ L^2[x]=a^4\{F(n,5)+\varepsilon F(n,4)\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$L^r[x]=a^{2r}\{F(n,2r+1)+\varepsilon F(n,2r)\}$$

hence

$$\left.\begin{array}{l}Y_1(x)=x+\varepsilon\sum_{r=1}^{N+1}a^{2r}F(n,2r)+\sum_{r=1}^{N}a^{2r}F(n,2r+1)\\ Y_2(x)=1+\sum_{r=1}^{N+1}a^{2r}F(n,2r)\\ Y_3(x)=0\end{array}\right\} \qquad (iii)$$

and

$$Y(x)=c(\varepsilon)Y_1(x)+\{b(\varepsilon)-\varepsilon c(\varepsilon)\}Y_2(x) \qquad (iv)$$

is the required solution.

Let α be a positive constant and put $\alpha=m+\delta$, where $m=integer$, $0\leqq\delta<1$. Then, if $m=even=2k$ we get

$$\left.\begin{aligned} Y_1(\alpha)&=\alpha+\delta\sum_{r=1}^{k}a^{2r}{}_mC_{2r}+\sum_{r=1}^{k-1}a^{2r}{}_mC_{2r+1} \\ Y_2(\alpha)&=1+\sum_{r=1}^{k}a^{2r}{}_mC_{2r} \end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots \quad (\mathrm{v})$$

and these can be transformed as follows

$$\left.\begin{aligned} Y_1(\alpha)&=\frac{\delta}{2}\{(1+a)^m+(1-a)^m\}+\frac{1}{2a}\{(1+a)^m-(1-a)^m\} \\ Y_2(\alpha)&=\frac{1}{2}\{(1+a)^m+(1-a)^m\} \end{aligned}\right\}\cdots\cdots \quad (\mathrm{vi}).$$

We can derive same expressions as (vi) when $m=odd$, and can prove that (vi) are true when $m\geqq1$.

In the case of $m=0$, special consideration is necessary, that is, if $m=0$, from (iii) we have

$$Y_1(\alpha)=\delta,\quad Y_2(\alpha)=1$$

and the right hand sides of (vi) are

$$\frac{\delta}{2}\{(1+a)^m+(1-a)^m\}+\frac{1}{2a}\{(1+a)^m-(1-a)^m\}=\delta$$

$$\frac{1}{2}\{(1+a)^m+(1-a)^m\}=1$$

under the condition that $a\neq1$, but if $a=1$ the right hand sides of (vi) are unknown. Now consider two functions $V_1(x)$, $V_2(x)$ such that

when $a\neq1$

$$V_1(x)=\frac{\varepsilon}{2}\{(1+a)^{x-\varepsilon}+(1-a)^{x-\varepsilon}\}+\frac{1}{2a}\{(1+a)^{x-\varepsilon}-(1-a)^{x-\varepsilon}\}$$

$$V_2(x)=\frac{1}{2}\{(1+a)^{x-\varepsilon}+(1-a)^{x-\varepsilon}\}$$

when $a=1$

$$\left.\begin{aligned} V_1(x)&=\frac{1+\varepsilon}{2}\times2^{x-\varepsilon} \\ V_2(x)&=\frac{1}{2}\times2^{x-\varepsilon} \end{aligned}\right\}\text{for } x-\varepsilon\geqq1,\qquad \left.\begin{aligned} V_1(x)&=\varepsilon \\ V_2(x)&=1 \end{aligned}\right\}\text{for } x=\varepsilon \quad\cdots(\mathrm{vii})$$

then we have

$$Y_1(\alpha)=V_1(\alpha),\quad Y_2(\alpha)=V_2(\alpha)\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (\mathrm{viii})$$

for any value of α. Moreover it is easily proved that $V_1(x)$, $V_2(x)$ satisfy the equation (i) and

$$Y(x)=c(\varepsilon)V_1(x)+\{b(\varepsilon)-\varepsilon c(\varepsilon)\}V_2(x)\cdots\cdots\cdots\cdots\cdots \quad (\mathrm{ix})$$

is the solution of (i) satisfying the conditions (ii).

Next, *when* $a \neq 1$, putting to

$$\left.\begin{aligned} b(\varepsilon)+\frac{1}{a}\,c(\varepsilon)&=2D(\varepsilon)(1+a)^{\varepsilon} \\ b(\varepsilon)-\frac{1}{a}\,c(\varepsilon)&=2E(\varepsilon)(1-a)^{\varepsilon} \end{aligned}\right\} \cdots\cdots\cdots\cdots \quad (\mathrm{x})$$

in equation (ix), we get

$$Y(x)=D(\varepsilon)(1+a)^{x}+E(\varepsilon)(1-a)^{x} \cdots\cdots\cdots\cdots \quad (\mathrm{xi}).$$

and this is well known result.

When $a=1$, putting to

$$b(\varepsilon)=D_1(\varepsilon)\cdot 2^{\varepsilon}, \quad c(\varepsilon)=E_1(\varepsilon)\cdot 2^{\varepsilon} \cdots\cdots\cdots\cdots \quad (\mathrm{xii})$$

we get

$$\left.\begin{aligned} Y(x)&=\frac{1}{2}\{D_1(\varepsilon)+E_1(\varepsilon)\}\times 2^{x} \cdots\cdots \text{ for } 1\leq x \\ Y(x)&=D_1(\varepsilon)\times 2^{x} \cdots\cdots\cdots\cdots \text{ for } x<1 \end{aligned}\right\} \cdots\cdots\cdots \quad (\mathrm{xiii}).$$

§ 7. Investigation of the continuity of the solution

Generally, the solution (17) may be a discontinuous function of x. We will investigate some conditions for the continuity of the solution under the following conditions.

(i) $b(x)$, $c(x)$ are continuous functions of x in the interval $0\leq x\leq 1$.

(ii) $p(x)$, $q(x)$ are continuous functions of x in the interval $0\leq x\leq 2N+2$.

Since $x=n+\varepsilon$, $0\leq\varepsilon<1$, $n=$ *integer*, the value of x at which $L^r[1]$ or $L^r[x]$ is discontinuous must be integer if they exists, and similarly the same statement comes into existence on the discontinuity of $Y_1(x)$, $Y_2(x)$ and $Y_3(x)$.

Accordingly the conditions that $Y(x)$ is continuous at $x=n$, $0\leq n\leq 2N+2$ are as follows:

$$\lim_{\varepsilon\to+0} Y(n+\varepsilon)=Y(n) \cdots\cdots\cdots\cdots \quad (\mathrm{i})$$

$$\lim_{\varepsilon\to 1-0} Y(n-1+\varepsilon)=Y(n) \cdots\cdots\cdots\cdots \quad (\mathrm{ii}).$$

By the previous conditions, it is evident that (i) is always satisfied, and from (ii), we get

$$c(1)\lim_{\varepsilon\to 1-0} Y_1(n-1+\varepsilon)+\{b(1)-c(1)\}\lim_{\varepsilon\to 1-0} Y_2(n-1+\varepsilon)+\lim_{\varepsilon\to 1-0} Y_3(n-1+\varepsilon)$$
$$=c(0)Y_1(n)+b(0)Y_2(n)+Y_3(n) \cdots\cdots \quad (22).$$
$$(1\leq n\leq 2N)$$

Specially for $n=1$ we get

$$b(1)=b(0)+c(0) \cdots\cdots\cdots\cdots \quad (23).$$

In the following, we particularily examine the continuity of $L^r[f(x)]$.
Put $K_r(x)=L^r[f(x)]$, then we get

$$K_1(x)=\mathop{S}\limits_{\varepsilon}^{x}\left\{\mathop{S}\limits_{\varepsilon}^{x}p(x)f(x)\Delta x\right\}\Delta x=\mathop{S}\limits_{\varepsilon}^{x-1}p(x)f(x)\Delta x+\mathop{S}\limits_{\varepsilon}^{x-2}p(x)f(x)\Delta x+\cdots\cdots$$

$$\cdots\cdots\cdots\cdots+\mathop{S}\limits_{\varepsilon}^{1+\varepsilon}p(x)f(x)\Delta x+\mathop{S}\limits_{\varepsilon}^{\varepsilon}p(x)f(x)\Delta x$$

$$=p(x-2)f(x-2)+2p(x-3)f(x-3)+\cdots\cdots+(x-1-\varepsilon)p(\varepsilon)f(\varepsilon),\ n\geqq 2$$

that is

$$K_1(x)=L[f(x)]=p(n-2+\varepsilon)f(n-2+\varepsilon)+2p(n-3+\varepsilon)f(n-3+\varepsilon)+\cdots$$
$$\cdots\cdots\cdots+(n-2)p(1+\varepsilon)f(1+\varepsilon)+(n-1)p(\varepsilon)f(\varepsilon),\ n\geqq 2\cdots\cdots\cdots \quad (24),$$

therefore

$$\lim_{\varepsilon\to+0} K_1(n+\varepsilon)=K_1(n)=p(n-2)f(n-2)+2p(n-3)f(n-3)+\cdots\cdots$$
$$\cdots\cdots\cdots+(n-2)p(1)f(1)+(n-1)p(0)f(0),$$

$$\lim_{\varepsilon\to1-0} K_1(n-1+\varepsilon)=p(n-2)f(n-2)+2p(n-3)f(n-3)+\cdots\cdots$$
$$\cdots\cdots\cdots+(n-3)p(2)f(2)+(n-2)p(1)f(1),$$

hence

$$K_1(n)-\lim_{\varepsilon\to1-0} K_1(n-1+\varepsilon)=(n-1)p(0)f(0)\cdots\cdots\cdots\cdots \quad (25).$$

Therefore the condition that $K_1(x)$ is continuous at $x=n$ is

$$p(0)f(0)=0\cdots\cdots\cdots\cdots \quad (26)$$

and this is independent of n, consequently in this case $K_1(x)$ is continuous everywhere.
Next, $K_2(x)=L[K_1(x)]$ and $K_1(\varepsilon)=\mathop{S}\limits_{\varepsilon}^{\varepsilon}\left\{\mathop{S}\limits_{\varepsilon}^{x}p(x)f(x)\Delta x\right\}\Delta x=0$ therefore $K_1(0)=0$, moreover $K_1(x)$ is continuous everywhere if (26) is satisfied. Hence we can assert that $K_2(x)$ is continuous everywhere considering the above argument taking $K_1(x)$ in place of $f(x)$. Quite similarly we can prove the continuity of $L^3[f(x)]$, $L^4[f(x)]$, $\cdots\cdots$ $L^r[f(x)]$ under the condition (26).
Now, concerning $Y_1(x)$ we have $f(x)=x$, hence $f(0)=0$, therefore (26) is satisfied and $Y_1(x)$ is continuous everywhere. While for $Y_2(x)$ we have $f(x)=1$, hence $f(0)=1\neq 0$, therefore the continuity of $Y_2(x)$ is unknown*.

For $Y_3(x)$ we have $f(x)=Q(x)$ and $Q(0)=0$ but the continuity of $Q(x)$ is unknown, therefore the continuity of $Y_3(x)$ is also unknown.

If $p(0)=0$ then the condition (26) is satisfied independently of $f(x)$ and moreover $Q(x)$ is continuous, hence $Y_1(x)$, $Y_2(x)$ and $Y_3(x)$ are all continuous.
For the example discussed in § 6, in the case of $a\neq 1$, we get

* $L[1]$, $L^2[1]$, $\cdots\cdots$ are discontinuous, but we can not assert that their sum is always discontinuous.

$$Y_1(m)=\frac{1}{2a}\{(1+a)^m-(1-a)^m\}$$

$$\lim_{\delta\to 1-0} Y_1(m-1+\delta)=\frac{1}{2}\{(1+a)^{m-1}+(1-a)^{m-1}\}+\frac{1}{2a}(1+a)^{m-1}-(1-a)^{m-1}\}$$

$$=\frac{1}{2a}\{(1+a)^m-(1-a)^m\}=Y_1(m)$$

$$Y_2(m)=\frac{1}{2}\{(1+a)^m+(1-a)^m\}$$

$$\lim_{\delta\to 1-0} Y_2(m-1+\delta)=\frac{1}{2}\{(1+a)^{m-1}+(1-a)^{m-1}\}\neq Y_2(m).$$

From the condition (22), we get

$$\left\{\frac{1}{a}c(1)+\frac{b(1)-c(1)}{1+a}-\frac{1}{a}c(0)-b(0)\right\}(1+a)^m$$
$$+\left\{\frac{-1}{a}c(1)+\frac{b(1)-c(1)}{1-a}+\frac{1}{a}c(0)-b(0)\right\}(1-a)^m=0,$$

and in order to that this relation is satisfied independently of $(1+a)^m$ and $(1-a)^m$ their coefficients must be zero, and we get

$$\frac{1}{1+a}\left\{b(1)+\frac{1}{a}c(1)\right\}=b(0)+\frac{1}{a}c(0),\ \frac{1}{1-a}\left\{b(1)-\frac{1}{a}c(1)\right\}=b(0)-\frac{1}{a}c(0)$$

this shows that $D(1)=D(0)$, $E(1)=E(0)$, and in this case, $Y(x)$ expressed by (xi) is everywhere continuous.

§8. Concluding words

In this discussion, $p(x)$ and $q(x)$ have no condition for their forms except their boundness, hence the function $Y(x)$ *given by* (15) *is the solution of the second order linear difference equation in general.*

The function $Y(x)$ is the accurate solution analytically, but in practice, successive application of the operator L is slightly troublesome as the example shows, and to find the numerical values of the solution requires considerable labour and it seems somewhat non-practical.

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ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTIONS OF SIMULTANEOUS DIFFERENCE EQUATIONS

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§ 1. Introduction

As is well known, the existence and uniqueness of the solutions of simultaneous *differential* equations have been proved under some conditions such that boundness of given functions, Lipschitz conditions and initial conditions.

Concerning simultaneous *difference* equations, our study shows that the existence and uniqueness of their solutions can be proved under similar conditions as above, and the method of proof is almost the same but there is a little difference about the conditions.

Let $y(x)$, $z(x)$ be two unknown functions, and as usual, put to

$$\Delta x=1,\ y_x=y(x),\ \Delta y_x=y(x+1)-y(x),\ \text{etc.}.$$

Moreover we can put to $0\leqq x$ with no loss of generality.

Now consider following simultaneous difference equations with respect to $y(x)$, $z(x)$:

$$\left.\begin{aligned}\Delta y_x&=f(x,\ y_x,\ z_x)\\ \Delta z_x&=\varphi(x,\ y_x,\ z_x)\end{aligned}\right\}\quad 0\leqq x\leqq r \qquad \cdots\cdots\cdots (1)$$

where $f(x,y,z)$ and $\varphi(x,y,z)$ are two given functions of x, y and z.

In the following by taking (1) as a general form of the first order simultaneous difference equations with respect to two unknown functions, we will discuss the existence and uniqueness of their solutions under some conditions.

§ 2. Conditions

Let $[x]=n$ be the integer part of x, and put to $\varepsilon=x-n$, then ε denotes the decimal part of x and $0\leqq\varepsilon<1$.

Now we give following conditions.

(i) **Boundness of $f(x,y,z)$ and $\varphi(x,y,z)$.**

Let $b(x)$, $c(x)$ be two given functions of x in the interval $0\leqq x\leqq 1$, and

$\alpha\leqq b(x)\leqq\beta\quad \alpha'\leqq c(x)\leqq\beta'$ for $0\leqq x\leqq 1$,

where $\alpha,\beta,\alpha',\beta'=$constants.

Next let ρ be a positive constant and consider a domain G such that the coordinates (x,y,z) of any point in G satisfy

$$0 \leqq x \leqq r+1,\ \alpha-\rho \leqq y \leqq \beta+\rho,\ \alpha'-\rho \leqq z \leqq \beta'+\rho \quad \cdots\cdots\cdots \quad (2),$$

and those of outside G does not satisfy (2).

Then it is easily proved that if x, y, z satisfy the ineaqualities

$$0 \leqq x \leqq r+1,\ |y-b(\varepsilon)| \leqq \rho,\ |z-c(\varepsilon)| \leqq \rho \quad \cdots\cdots\cdots \quad (3),$$

then the point (x, y, z) lies in G.

Now let $f(x, y, z)$, $\varphi(x, y, z)$ be bounded for x, y, z which are the coordinates of any point in G, and suppose that there exists M such that

$$|f(x, y, z)| \leqq M,\ |\varphi(x, y, z)| \leqq M \quad \cdots\cdots\cdots \quad (4),$$

moreover let $r+1 \leqq \rho/M$.*

(ii) **Lipschitz conditions.**

Let $f(x, y, z)$, $\varphi(x, y, z)$ satisfy following Lipschitz conditions:

$$\left.\begin{aligned} |f(x_1, y_1, z_1)-f(x_2, y_2, z_2)| &\leqq A|y_1-y_2|+B|z_1-z_2| \\ |\varphi(x_1, y_1, z_1)-\varphi(x_2, y_2, z_2)| &\leqq A|y_1-y_2|+B|z_1-z_2| \end{aligned}\right\} \quad (5)$$

for any two points (x_1, y_1, z_1), (x_2, y_2, z_2) in G, where $A, B=$ positive constants.

(iii) **Initial conditions.**

When $0 \leqq x < 1$, that is $x=\varepsilon$, let $y(x)$, $z(x)$ satisfy

$$y_x=b(\varepsilon),\ z_x=c(\varepsilon) \quad \cdots\cdots\cdots \quad (6).$$

In the following, we will prove the existence and uniqueness of the solutions of (1) under the conditions (i), (ii) and (iii) and show that $\big(x, y(x), z(x)\big)$ lies in the domain G for $0 \leqq x \leqq r+1$.

§ 3. Proof of the existence theorem

In the first place, if $y(x)$, $z(x)$ satisfy (1) and (6) then we have

$$\left.\begin{aligned} y_x &= b(\varepsilon)+\mathfrak{S}_0^x f(x, y_x, z_x)\Delta x \\ z_x &= c(\varepsilon)+\mathfrak{S}_0^x \varphi(x, y_x, z_x)\Delta x \end{aligned}\right\} \quad (7)$$

Inversely we can obtain (1) and (6) from (7), therefore (7) is equivalent to (1) and (6).**

---

* If $\rho/M < r+1$, we put to $0 \leqq x \leqq \frac{\rho}{M}$ instead of $0 \leqq x \leqq r+1$ anew, that is $\frac{\rho}{M}-1$ instead of r.

** For any function $F(x)$, we have $\Delta F(x-[x])=0$, $\Delta \mathfrak{S}^x F(x)\Delta x=F(x)$.

Next put

$$\left.\begin{aligned}
&\Phi_0(x)=b(x-[x])=b(\varepsilon),\ \Psi_0(x)=c(x-[x])=c(\varepsilon),\\
&\Phi_{k+1}(x)=b(\varepsilon)+\int_{\varepsilon}^{x} f\big(x,\ \Phi_k(x),\ \Psi_k(x)\big)\varDelta x\\
&\Psi_{k+1}(x)=c(\varepsilon)+\int_{\varepsilon}^{x} \varphi\big(x,\ \Phi_k(x),\ \Psi_k(x)\big)\varDelta x\\
&\qquad\qquad 0\leqq x\leqq r+1,\ 0\leqq k
\end{aligned}\right\}\ \cdots\cdots\cdots\cdots\cdots\ (8).$$

and suppose that

$$\left.\begin{aligned}
|\Phi_k(x)-b(\varepsilon)|\leqq Mn\\
|\Psi_k(x)-c(\varepsilon)|\leqq Mn
\end{aligned}\right\}\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ (9),$$

then, since $n\leqq x\leqq r+1\leqq \dfrac{\rho}{M}$, we have $Mn\leqq\rho$, hence $\big(x,\ \Phi_k(x),\ \Psi_k(x)\big)$ lies in G, therefore

$$\big|f\big(x,\ \Phi_k(x),\ \Psi_k(x)\big)\big|\leqq M,\quad \big|\varphi\big(x,\ \Phi_k(x),\ \Psi_k(x)\big)\big|\leqq M.$$

Accordingly, from (8), we get

$$|\Phi_{k+1}(x)-b(\varepsilon)|\leqq\int_{\varepsilon}^{x}\big|f\big(x,\ \Phi_k(x),\ \Psi_k(x)\big)\big|\varDelta x\leqq Mn\leqq\rho$$

$$|\Psi_{k+1}(x)-c(\varepsilon)|\leqq\int_{\varepsilon}^{x}\big|\varphi\big(x,\ \Phi_k(x),\ \Psi_k(x)\big)\big|\varDelta x\leqq Mn\leqq\rho,$$

that is, (9) are also satisfied for $k+1$, while (9) are actually satisfied for $k=0$, hence (9) are satisfied for any value of k. Therefore $\big(x,\ \Phi_k(x),\ \Psi_k(x)\big)$ lies in G for any value of k, and we get

$$\begin{aligned}
|\Phi_{k+1}(x)-\Phi_k(x)|&\leqq\int_{\varepsilon}^{x}\big|f\big((x,\Phi_k(x),\Psi_k(x)\big)-f\big(x,\Phi_{k-1}(x),\ \Psi_{k-1}(x)\big)\big|\varDelta x\\
&\leqq\int_{\varepsilon}^{x}\Big\{A|\Phi_k(x)-\Phi_{k-1}(x)|+B|\Psi_k(x)-\Psi_{k-1}(x)|\Big\}\varDelta x,
\end{aligned}$$

similarly we have

$$\left.\begin{aligned}
|\Phi_{k+1}(x)-\Phi_k(x)|\leqq\int_{\varepsilon}^{x}\Big\{A|\Phi_k(x)-\Phi_{k-1}(x)|+B|\Psi_k(x)-\Psi_{k-1}(x)|\Big\}\varDelta x\\
|\Psi_{k+1}(x)-\Psi_k(x)|\leqq\int_{\varepsilon}^{x}\Big\{A|\Phi_k(x)-\Phi_{k-1}(x)|+B|\Psi_k(x)-\Psi_{k-1}(x)|\Big\}\varDelta x
\end{aligned}\right\}\cdots\ (10).$$

Now suppose that

$$\left.\begin{aligned}|\varPhi_k(x)-\varPhi_{k-1}(x)|&\leqq\frac{M}{k!}(A+B)^{k-1}n^k\\|\varPsi_k(x)-\varPsi_{k-1}(x)|&\leqq\frac{M}{k!}(A+B)^{k-1}n^k\end{aligned}\right\}\cdots\cdots\cdots\cdots\cdots\cdots(11),$$

then from (10) we get

$$|\varPhi_{k+1}(x)-\varPhi_k(x)|\leqq\frac{M}{k!}(A+B)^k\int_\varepsilon^x n^k\varDelta x\leqq\frac{M}{k!}(A+B)^k\int_0^n n^k\varDelta n=\frac{M(A+B)^k}{(k+1)!}\cdot n^{k+1},$$

similarly

$$|\varPsi_{k+1}(x)-\varPsi_k(x)|\leqq\frac{M}{(k+1)!}(A+B)^k n^{k+1}.$$

While we can easily prove that (11) are actually satisfied for $k=1$, therefore (11) are satisfied for any value of k.

Here, $\sum_{k=1}^{\infty}\frac{M}{k!}(A+B)^{k-1}n^k$ absolutely converges to $\frac{M}{A+B}[exp\{(A+B)n\}-1]$ for any finite value of x (therefore n). Accordingly

$$\sum_{k=1}^{\infty}\{\varPhi_{k+1}(x)-\varPhi_k(x)\}\ \text{and}\ \sum_{k=1}^{\infty}\{\varPsi_{k+1}(x)-\varPsi_k(x)\}$$

absolutely and uniformly converge, hence $\varPhi_k(x)$ and $\varPsi_k(x)$ also uniformly converge. Therefore, there exist definite functions $\varPhi(x)$, $\varPsi(x)$ such that

$$\lim_{k\to\infty}\varPhi_k(x)=\varPhi(x),\quad \lim_{k\to\infty}\varPsi_k(x)=\varPsi(x).$$

Hence from (8), puttong $k\to\infty$, we have

$$\varPhi(x)=b(\varepsilon)+\int_\varepsilon^x f\big(x,\varPhi(x),\varPsi(x)\big)\varDelta x$$

$$\varPsi(x)=c(\varepsilon)+\int_\varepsilon^x \varphi\big(x,\varPhi(x),\varPsi(x)\big)\big)\varDelta x.$$

This shows that $\varPhi(x),\varPsi(x)$ are the solutions of (1) satisfying (6) and also $\big(x,\varPhi(x),\varPsi(x)\big)$ lies in the domain G.

§4. Uniqueness of the solutions

Suppose that $\varPhi(x)$, $\varPsi(x)$ and $\tilde{\varPhi}(x)$, $\tilde{\varPsi}(x)$ are two sets of the solutions of (7) for $0\leqq x\leqq r+1$ and $\big(x,\varPhi(x),\varPsi(x)\big)$ and $\big(x,\tilde{\varPhi}(x),\tilde{\varPsi}(x)\big)$ lie in G, then we have

$$\Phi(x)=b(\xi)+\int_{\xi}^{x} f(x,\Phi,\Psi)\Delta x, \qquad \tilde{\Phi}(x)=b(\xi)+\int_{\xi}^{x} f(x,\tilde{\Phi},\tilde{\Psi})\Delta x$$

$$\therefore \quad |\Phi(x)-\tilde{\Phi}(x)|\leqq\int_{\xi}^{x}\{A|\Phi(x)-\tilde{\Phi}(x)|+B|\Psi(x)-\tilde{\Psi}(x)|\}\Delta x$$

similarly

$$|\Psi(x)-\tilde{\Psi}(x)|\leqq\int_{\xi}^{x}\{A|\Phi(x)-\tilde{\Phi}(x)|+B|\Psi(x)-\tilde{\Psi}(x)|\}\Delta x \qquad \cdots \ (12).$$

Put

$$H(x)=max\{|\Phi(x)-\tilde{\Phi}(x)|,\ |\Psi(x)-\tilde{\Psi}(x)|\} \cdots\cdots (13)$$

then

$$|\Phi(x)-\tilde{\Phi}(x)|\leqq(A+B)\int_{\xi}^{x}H(x)\Delta x,\ |\Psi(x)-\tilde{\Psi}(x)|\leqq(A+B)\int_{\xi}^{x}H(x)\Delta x,$$

therefore

$$0\leqq H(x)\leqq(A+B)\int_{\xi}^{x}H(x)\Delta x \cdots\cdots (14).$$

Evidently there exists a constant K such that $H(x)\leqq K$, hence from (14) we get

$$H(x)\leqq(A+B)K\int_{\xi}^{x}\Delta x=(A+B)Kn$$

$$\therefore \quad H(x)\leqq(A+B)^2K\int_{\xi}^{x}n\Delta x=(A+B)^2K\int^{n}n\Delta n\leqq\frac{(A+B)^2n^2}{2}K$$

$$\therefore \quad H(x)\leqq(A+B)^3K\int_{\xi}^{x}\frac{n^2}{2}\Delta x=(A+B)^3K\int^{n}\frac{n^2}{2}\Delta n\leqq\frac{(A+B)^3n^3}{3!}K$$

....

....

generally

$$H(x)\leqq\frac{1}{k!}(A+B)^kn^kK \quad \text{for any } k,$$

while

$$\lim_{k\to\infty}\frac{1}{k!}(A+B)^kn^k=0$$

therefore $H(x)\equiv0$ that is $\Phi(x)\equiv\tilde{\Phi}(x)$, $\Psi(x)\equiv\tilde{\Psi}(x)$,
this proves the uniqueness.

§5. Concluding words

On the outline of the existence and uniqueness of the solutions, we can elementarily understand by following consideration, and also can understand that why the conditions $y(\varepsilon)=b(\varepsilon)$, $z(\varepsilon)=c(\varepsilon)$ must be taken instead of the conditions $y(0)=b$, $z(0)=c$.

From (1) we get

$$\left.\begin{aligned} y_{x+1}&=y_x+f(x,\ y_x,\ z_x)\\ z_{x+1}&=z_x+\varphi(x,\ y_x,\ z_x)\end{aligned}\right\} \qquad (1)'$$

When $0\leqq x<1$, since $x=\varepsilon$ and $y(x)=b(\varepsilon)$, $z(x)=c(\varepsilon)$, from (i)', we get

$$\left.\begin{aligned} y_{x+1}&=b(\varepsilon)+f\big(\varepsilon,\ b(\varepsilon),\ c(\varepsilon)\big)\\ z_{x+1}&=c(\varepsilon)+\varphi\big(\varepsilon,\ b(\varepsilon),\ c(\varepsilon)\big)\end{aligned}\right\} \text{ where } 1\leqq x+1<2$$

therefore the values of y_x, z_x for $1\leqq x<2$ are determined. Similarly when $1\leqq x<2$, the values of y_x, z_x for $2\leqq x<3$ are determined, and by repeating this procedure we can uniquely determine the values of y_x, z_x for $n\leqq x<n+1$, where n is any positive integer provided that the functions $f(x,y,z)$, $\varphi(x,y,z)$ are one valued and bounded for any values of x, y, z in some domain.

The functions $y(x)$, $z(x)$ which are determined by the above procedure may have discontinuous points which must correspond to that $x=integer$ if they exist, even if $b(\varepsilon)$, $c(\varepsilon)$ are continuous for $0\leqq\varepsilon<1$ and $f(x,y,z)$, $\varphi(x,y,z)$ are both continuous for any (x,y,z) in some domain.

For example, at $x=m=integer$ we have

$$\lim_{\varepsilon\to 0} y_{m+\varepsilon}=y_m,$$

therefore if

$$\Delta_m=y_m-\lim_{\varepsilon\to 0} y_{m-1+\varepsilon}\neq 0$$

y_x is discontinuous at $x=m$ and has a jump Δ_m.

It is easy to extend the above argument for many number of unknown functions.

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CHARACTERS OF THE ELECTROMAGNETIC AND STRONG MESONIC INTERACTIONS

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Summary

As is the case for quantum elecrodynamics (Q. E. D.), there exists the gauge-like transformations under which the theory is invariant for the pion-nucleon system. Provided that the strong interactions containing pions are all charge independent, it is found that the existence of such kind of the invariance is possible when the assumption of the "*global symmetry*" for these interactions is correct. If we require, for all of strong interactions between pions and baryons, the invariance under newly proposed extended gauge-like transformations besides the well known characters such as proper Lorentz invariance, charge independence, conservations of charge, strangeness and baryon number, etc., we will naturally obtain the P-, C- and T- invariant theory as well as the idea of the "global symmetry". It is also found that the forms of the interactions mentioned above are all the derivative type. The "*principle of the minimal electromagnetic interaction*" is derived as one of the results of our stand point of view. The essential parts of this paper are already published in the other place[1]. In this paper, the ideas and the methods developed there are treated in more detail and further extended to the vector coupling case which is not treated previously.

§ . 1 Introduction

In the previous paper[2)*], the properties of the theory under P- and T-transformations have been studied for generalized Q. E. D. Firstly, by assuming the invariance only under the proper Lorentz transformation, we generalized the Dirac equation, and then, using this generalized Dirac equation, we constructed Q. E. D. for the fermions in the general way. As we have assumed only the proper Lorentz invariance, the forms of the interactions should be so chosen that they may have the most general forms which are admittable within the above mentioned assumption. But, as was shown in I, even though we construct Q. E. D. along the line stated above, if the type of the interaction is vectorial, the theory is found to have no different point from the ordinary one which is built so as to be also invariant under the improper Lorentz transformation. On the contrary, if we introduce the Pauli-type interaction in its most general form, we can not remove the factor which is not invariant under the improper Lorentz transformation from this generalized Pauli-term. As this factor exhibits the non-invariant effects under P- and T-transformations, if it is true that the electromagnetic phenomena are P- and T-invariant, we must get rid of the Pauli-term from Q. E. D. using some kind of the selection rule.

* In the following, we will cite the 2nd paper of the reference 2) as I.

So far, the invariances of the theories under each of the P–, C– and T–transformations seem to be undoubtful for the electromagnetic phenomena as well as the ones in which the strong interactions are playing roles. But, as for the weak interactions, we must accept the fact that the above invariances do not hold anymore. Indeed, there are many experiments (the experiments concerning the disintegration of the elementary particles) which shows the violence of these invariances. To understand the strong and the weak interactions unificatively, it seems to be reasonable to consider that all of the interactions are originally not invariant under P– and T–transformations and that the strong interactions are distinguished from the weak ones by some additional conservation law or something similar.

In 1957, H. S. Green and C. A. Hurst[3] showed that the violence of P–invariance in the weak interactions can be understood as that the fermions have the mixed parities. In I, we have shown that to generalize the Dirac equation is equivalent to consider the parity mixture for the fermion wave function. Further, it has been found that there is a suitable transformation* by which the generalized Dirac equation goes thoroughly back to the ordinary one (in the following, we will call this transformation "reducing transformation"** and will write it simply as R. T.). As the phenomena relating to the strong interactions do not show the violences of P– and T–invariances,[4] the generalized theories for them should not have any effective difference to the ordinary theories in which the invariance under the improper Lorentz transformation is also taken into account. In order words, the quantum electrodynamics and the theories for the strong interactions must entirely be independent of the mixing ratio of parity for the fermion participating the phenomenon in all of their aspects. If we stand on this point of view, the weak interactions are naturally introduced into our understanding of the nature.

Of course, it is difficult to give the general discussions about the reason why P–, C– and T–invariances must hold separately in the strong interactions, but, for Q. E. D., when only the vectorial interaction is present, we could say that this is due to the fact that it must obey one more conservation law (the charge-current conservation law) than the weak interactions or must have one more invariant property (the property of the gauge invariant) than the latter. Among the other strong interactions, since the one containing the boson with strangeness zero are pretty analogous to Q. E. D., we will examine in the following the character of the strong interactions of the Yukawa type between the baryon and the boson with strangeness zero, especially the pion, taking Q. E. D. as a clue.

In order to obtain the selection rule satisfactory, we have extended the concept of the gauge transformation rather artificially. The Pauli-term can be treated separately from the vectorial-term and is found to be abandoned from our stand point of view. The selection rule obtained can also be applicable for all of the iso-bose bosons of the strangeness zero. The strong interactions between the baryons and the K–particle have quite the different characters from the strong mesonic interactions, and could not treated in parallel with the cases of the pions.

When the requirement of the invariance under the new kind of the gauge-like

* The formula (20) in I.
** "returning transformation" in I.

transformations is applied to the free boson part of the Lagrangian, it is necessary to introduce the artificial device, but the physical meaning of it is not yet clear.

If we write the generalized Dirac equation in the form (the formula (9) in I)

$$(e^{A\gamma_5}\gamma_\mu\partial_\mu+\kappa e^{-B\gamma_5})\psi=0, \qquad (1)$$

the rate of mixing the parity for the fermion wave function can be determined from the values of two constants A and B in the above equation. These constants are quite arbitrary in (1). Now, though it is one of the unsolved questions whether the assumption of the "global symmetry"[5] holds for the strong mesonic interactions, if we can give the values of the above constants arbitrary and differently to the different fermions, we can say that the strong mesonic interactions should be globally symmetric or rather, at least the interactions $\Lambda\Sigma\pi$ and $\Sigma\Sigma\pi$ are of the same strength. This is also necessary to retain the invariance under the extended gauge-like transformations.

In general, for the requirement of the invariance under the extended gauge-like transformations, there corresponds one identical relation such as the relation of the charge-current conservation in Q. E. D.. The relation thus obtained must be inspected by the physical experiences. If this relation contradict with the actual phenomena, one must abandon the theory which draws out this unreasonable relation. In this way, the possibility of the photon wave function being pseudovector must be excluded.

In the section 2 of this paper, we propose the selection rule to remove the Pauli-term from Q. E. D. by introducing the extended gauge transformations. We will treat the case in which the wave function of the photon is pseudovector (pseudo Q. E. D.) in the section 3. The section 4 is devoted to the discussions about the strong mesonic interactions and about the strong interactions between the baryon and the iso-bose boson with strangeness zero.

Some comments on the strong K interactions and the comparison with the selection rule obtained by S. Ozaki[6] will be given in the section 5.

§ . 2 Quantum Electrodynamics

In general, the total Lagrangian for the system of the nucleon and the electromagnetic field can be written as[7]

$$\left.\begin{aligned}
&L=L_N+L_{int}+L_R,\quad L_{int}=L_v+L_p,\\
&L_N=-\int\bar{N}(e^{A\gamma_5}\gamma_\mu\partial_\mu+\kappa e^{-B\gamma_5})Nd^4x,\\
&L_v=-ie\int\bar{N}e^{A\gamma_5}e^{-\delta\gamma_5}\gamma_\mu A_\mu\frac{1}{2}(1+\tau_3)Nd^4x,\\
&L_p=\frac{1}{2}(e'/\kappa)\int Ne^{k\gamma_5}\sigma_{\mu\nu}F_{\mu\nu}\frac{1}{2}(1+\tau_3)Nd^4x,\\
&L_R=-\frac{1}{4}\int F_{\mu\nu}F_{\mu\nu}d^4x
\end{aligned}\right\} \qquad (2)$$

where*

* The expression (16) in I.

$$\bar{N}=N^{*}exp\left(\frac{1}{2}\left\{(B+B^{*})-(A-A^{*})\right\}\gamma\right),\quad N^{*}=N^{\dagger}\gamma_{4},$$

and N is the nucleon wave function of the one column and 8 rows obeying the generalize Dirac equation and N^{*} is its hermitian conjugate.

From the hermiticity condition, we get the relations*

$$\delta=\delta^{*},\ k=-B+is\ (s:\ real\ constant).$$

Further, from the charge-current conservation law

$$\partial_{\mu}j_{\mu}=0, \quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\quad (3)$$

$$j_{\mu}=ie\bar{N}e^{(\alpha-\alpha^{*})\gamma_5}\gamma_{\mu}\frac{1}{2}(1+\tau_3)e^{s\gamma_5}N, \quad\cdots\cdots\cdots\cdots\cdots\cdots\quad (4)$$

$$\left(\alpha=\frac{1}{2}(A+B)\right),$$

we have $\delta=0$.** As this conservation law has no effect on the Pauli-term, we can not get rid of the factor exhibiting the P- and T-non-invariant effect (e^{is}) from it. Thus, the difference appears between the vectorial and the Pauli-type interactions corresponding to the different characters concerning to the charge-current conservation, and this difference is the very footing by which the Pauli-term can be get rid of from Q. E. D. The same result is able to be obtained[2)] by requiring that the Lagrangian (2) is invariant under the gauge-transformations

$$\left.\begin{aligned} &A_{\mu}\to A'_{\mu}=A_{\mu}+\partial_{\mu}\Lambda,\\ &N\to N'=e^{\frac{1}{2}ie\Lambda(1+\tau_3)}N. \end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots\quad (5)$$

That is to say, in Q. E. D., there exists the intimate relationship between the condition $\partial_{\mu}j_{\mu}=0$ and the requirement of the gauge-invariance. The property of the gauge-invariance comes essentially from the facts that the photon has not mass and its wave function A_{μ} is a vector. If A_{μ} is not a vector, this relationship does not exist anymore. (c. f. §. 3)

Now, the condition $\delta=0$ can be obtained alternatively in the following manner. Consider the apparent gauge-like transformations

$$\left.\begin{aligned} &\frac{1}{2}(1+\tau_3)A_{\mu}\to\frac{1}{2}(1+\tau_3)A_{\mu}+I\cdot\partial_{\mu}\Lambda,\\ &N\to e^{ie\Lambda}N, \end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots\quad (6)$$

where I denotes the unit matrix. We will call these transformations the "*extended gauge transformations*". The first part of these transformations is the one relating the iso-space to the ordinary space-time. It is obvious that L_N+L_{int} is invariant under the transformations (6) when and only when $\delta=0$. To show that L_R is also invariant under these transformations, we must contrive some artificial treatment. That is to say, by introducing the constant spinor of 8 components ϵ having the properties

* If we put $B=R_eB+iI_mB$, then the condition for k takes the form $k=-(1/2)(B+B^{*})+i(s-I_mB)$. If we put $s-I_mB$ newly s, then this condition is found to be the same as the formula (37) in I. That is, s in the text is equal to g in I.

** See Chapter III in I.

$$\bar{\epsilon}\,\frac{1}{2}(1+\tau_3)\,\epsilon=1,\quad \bar{\epsilon}I\epsilon=1, \qquad (7)$$

we will reconsider the L_R as follows:

$$\left.\begin{aligned} L_R&\to\bar{\epsilon}L_R\epsilon \\ &=\bar{\epsilon}\left(-\frac{1}{4}\int F_{\mu\nu}F_{\mu\nu}d^4x\right)\epsilon \\ &=\bar{\epsilon}\left\{-\frac{1}{4}\int\frac{1}{2}(1+\tau_3)F_{\mu\nu}\cdot\frac{1}{2}(1+\tau_3)F_{\mu\nu}d_4x\right\}\epsilon. \end{aligned}\right\} \qquad (8)$$

The expression (8) is easily found to be invariant under the transformations (6). There always exists such a constant spinor ϵ as above. Even though the physical meaning of ϵ is not yet clear, by introducing the constant spinor ϵ, we find that there are two transformations under Q. E. D. is invariant and both of them have the intimate relationship with the charge-current conservation law $\partial_\mu j_\mu=0$.

Here, we must pay attention to the fact that both of the ordinary gauge-transformations (5) and the extended ones (6) are powerless to restrict the value of the constant s. Therefore, to get P- and T-invariant theory, we should find out some difference between the vectorial and the Pauli-type interactions. To do so, in the first place, we will transform the generalized wave function N to the ordinary one $\overset{c}{N}$ by using the R. T. that is

$$N=e^{\alpha\gamma_5}\overset{c}{N}. \qquad (9)$$

Then, the Lagrangian of the system becomes

$$\left.\begin{aligned} &L=\overset{c}{L}_N+\overset{c}{L}_{int}+L_R,\quad \overset{c}{L}_{int}=\overset{c}{L}_v+\overset{c}{L}_p, \\ &\overset{c}{L}_N=-\int\overset{c}{N}{}^+(\gamma_\mu\partial_\mu+\kappa)\overset{c}{N}d^4x, \\ &\overset{c}{L}_v=ie\int\overset{c}{N}{}^+\gamma_\mu A_\mu\frac{1}{2}(1+\tau_3)\overset{c}{N}d^4x, \\ &\overset{c}{L}_p=-\frac{1}{2}(e'/\kappa)\int\overset{c}{N}{}^+e^{is\gamma_5}\sigma_{\mu\nu}F_{\mu\nu}\frac{1}{2}(1+\tau_3)\overset{c}{N}d_4x. \end{aligned}\right\} \qquad (10)$$

It is easily found that, if N is replaced by $\overset{c}{N}$ in the transformations (5) and (6), then these transformations also make the Lagrangian (10) invariant.

For a while, we will proceed with the Lagrangian (10). In the following, we consider the vectorial interaction and the Pauli-type one separately. When $e=0$ and $e'\neq0$, that is to say, when there is only the Pauli-term, corresponding to the transformations (6), let us introduce the transformations

$$\left.\begin{aligned} &\frac{1}{2}(1+\tau_3)\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\to\frac{1}{2}(1+\tau_3)\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}+i\gamma_\mu\partial_\mu\Lambda, \\ &\overset{c}{N}\to e^{-i(e'/\kappa)\Lambda}\overset{c}{N},\quad \Box\Lambda=0. \end{aligned}\right\} \qquad (11)$$

It is easily found that the expression $\overset{c}{L}_N+\overset{c}{L}_p$ is invariant under these transformations when and only when $s=0$. We can show that the free radiation part of the Lagrangian L_R is invariant under the transformations (11) in the same way as before. To prove

it, we will examine the relation between

$$\bar{\epsilon}\,\frac{1}{2}(1+\tau_3)\,\frac{1}{2}\,\sigma_{\mu\nu}F_{\mu\nu}\cdot\frac{1}{2}(1+\tau_3)\,\frac{1}{2}\,\sigma_{\lambda\rho}F_{\lambda\rho}\epsilon$$

and $F_{\mu\nu}F_{\mu\nu}$, where ϵ is the constant spinor satisfying the relation (7). Using the relation

$$\frac{1}{2}\,\sigma_{\mu\nu}F_{\mu\nu}=(\vec{\sigma}\cdot\vec{\boldsymbol{H}})-i\,(\vec{\alpha}\cdot\vec{\boldsymbol{E}}) \quad\cdots\cdots\cdots\cdots \quad (12)$$

and the relations between α and σ* the former expression can be calculated as follows:

$$\begin{aligned}
&\bar{\epsilon}\,\frac{1}{2}(1+\tau_3)\,\frac{1}{2}\,\sigma_{\mu\nu}F_{\mu\nu}\cdot\frac{1}{2}(1+\tau_3)\,\frac{1}{2}\,\sigma_{\lambda\rho}F_{\lambda\rho}\epsilon\\
&=\bar{\epsilon}\,\frac{1}{2}(1+\tau_3)\Big((\vec{\sigma}\cdot\vec{\boldsymbol{H}})-i(\vec{\alpha}\cdot\vec{\boldsymbol{E}})\Big)\Big((\vec{\sigma}\cdot\vec{\boldsymbol{H}})-i\,(\vec{\alpha}\cdot\vec{\boldsymbol{E}})\Big)\epsilon\\
&=\bar{\epsilon}\,\frac{1}{2}(1+\tau_3)\Big[(\boldsymbol{H}^2-\boldsymbol{E}^2)-i\gamma_5\Big\{(\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{H}})+(\vec{\boldsymbol{H}}\cdot\vec{\boldsymbol{E}})\Big\}\\
&+\Big(\alpha_1\Big\{(E_2H_3-H_3E_2)+(H_2E_3-E_3H_2)\Big\}\\
&+\alpha_2\Big\{(E_3H_1-H_1E_3)+(H_3E_1-E_1H_3)\Big\}\\
&+\alpha_3\Big\{(E_1H_2-H_2E_1)+(H_1E_2-E_2H_1)\Big\}\Big)\Big]\epsilon\\
&=(\boldsymbol{H}^2-\boldsymbol{E}^2)\bar{\epsilon}\,\frac{1}{2}(1+\tau_3)\,\epsilon\\
&-i\Big\{(\vec{\boldsymbol{E}}\cdot\vec{\boldsymbol{H}})+(\vec{\boldsymbol{H}}\cdot\vec{\boldsymbol{E}})\Big\}\,\bar{\epsilon}\,\frac{1}{2}(1+\tau_3)\gamma_5\epsilon\\
&+\Big([\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}]+[\vec{\boldsymbol{H}}\times\vec{\boldsymbol{E}}]\Big)\cdot\bar{\epsilon}\,\frac{1}{2}(1+\tau_3)\alpha\,\epsilon.
\end{aligned}$$

As $\vec{\boldsymbol{E}}$ is perpendicular to $\boldsymbol{H}$, the second term of the right side vanishes. The third term can be reduced to zero. This can be shown by taking the limiting procedure in the following way. As

$$\begin{aligned}
&(E_2H_3-H_3E_2)+(H_2E_3-E_3H_2)\\
&=(E_2\partial_1A_2-\partial_1A_2E_2)+(E_3\partial_1A_3-\partial_1A_3E_3)\\
&=\lim_{x'\to x}\frac{\partial}{\partial x_1'}\Big\{\big[E_2(x),\ A_2(x')\big]_-+\big[E_3(x),\ A_3(x')\big]_-\Big\}\\
&=-8\pi i\lim_{x'\to x}\frac{\partial}{\partial x_1'}\,\delta(x-x'),
\end{aligned}$$

we get

$$\begin{aligned}
&\int\Big([\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}]+[\vec{\boldsymbol{H}}\times\vec{\boldsymbol{E}}]\Big)d^3x\\
&=-8\pi i\lim_{x'\to x}\int\boldsymbol{\nabla}\,\delta(x-x')\,d^3x
\end{aligned}$$

* The connections between γ-matrices and σ and α matrices are listed in the Appendix.

$$=-8\pi i \lim_{x'\to x} \nabla' \int \delta(x-x')d^4x=0.$$

Therefore, we can replace L_R as follows:

$$L_R=-\frac{1}{4}\int F_{\mu\nu}F_{\mu\nu}d^4x$$
$$\to -\overline{\in}\frac{1}{8}\int \frac{1}{2}(1+\tau_3)\sigma_{\mu\nu}F_{\mu\nu}\cdot\sigma_{\lambda\rho}F_{\lambda\rho}d^4x\cdot\in$$

Applying the transformations (11), L_R can be calculated as

$$L_R\to -\overline{\in}\frac{1}{2}\int\left\{\frac{1}{2}(1+\tau_3)\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}+i\gamma_\mu\partial_\mu\Lambda\right\}\cdot$$
$$\left\{\frac{1}{2}(1+\tau_3)\frac{1}{2}\sigma_{\lambda\rho}F_{\lambda\rho}+i\gamma_\rho\partial_\rho\Lambda\right\}d^4x\cdot\in$$
$$=\frac{1}{2}\int(\boldsymbol{E}^2-\boldsymbol{H}^2)d^4x+\frac{1}{2}\overline{\in}\int\gamma_\mu\partial_\mu\Lambda\cdot\gamma_\nu\partial_\nu\Lambda d^4x\in$$
$$-\overline{\in}\frac{i}{2}\int\frac{1}{2}(1+\tau_3)\left\{\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\gamma_\rho\partial_\rho\Lambda+\gamma_\rho\partial_\rho\Lambda\cdot\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\right\}d^4x\in.$$

The second term of the right side vanishes using the condition on Λ, i.e., $\Box\Lambda=0$. The terms in the bracket in the third term of the right side can be calculated as

$$\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\cdot\gamma_\rho\partial_\rho\Lambda+\gamma_\rho\partial_\rho\Lambda\cdot\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$$
$$=(\vec{\sigma}\cdot\vec{\boldsymbol{H}}-i\vec{\alpha}\cdot\vec{\boldsymbol{E}})(\beta\partial_4\Lambda-i\beta\vec{\alpha}\nabla\Lambda)$$
$$+(\beta\partial_4\Lambda-i\beta\vec{\alpha}\nabla\Lambda)(\vec{\sigma}\cdot\vec{\boldsymbol{H}}-i\vec{\alpha}\cdot\vec{\boldsymbol{E}})$$
$$=2\beta\vec{\sigma}\cdot\vec{\boldsymbol{H}}\partial_4\Lambda+2i\beta\gamma_5\vec{\boldsymbol{H}}\cdot\nabla\Lambda+2i\beta\vec{\sigma}\cdot[\vec{\boldsymbol{E}}\times\nabla\Lambda].$$

Here, we used the relations

$$\sigma_i\alpha_k+\alpha_k\sigma_i=-2\gamma_5\delta_{ik},$$

and

$$(\vec{\alpha}\cdot\vec{\boldsymbol{A}})(\vec{\alpha}\cdot\vec{\boldsymbol{B}})-(\vec{\alpha}\cdot\vec{\boldsymbol{B}})(\vec{\alpha}\cdot\vec{\boldsymbol{A}})=2i\vec{\sigma}\cdot[\vec{\boldsymbol{A}}\times\vec{\boldsymbol{B}}],$$

when

$$[A_i,\ B_k]_-=0.$$

By integrating partially, as the second term vanishes because $div\vec{\boldsymbol{H}}=0$, we obtain

$$-(2\beta\vec{\sigma}\cdot\partial_4\vec{\boldsymbol{H}}-2\beta i\vec{\sigma}\cdot rot\vec{\boldsymbol{E}})\Lambda$$
$$=2i\beta\vec{\sigma}\left(\frac{\partial\dot{\boldsymbol{H}}}{\partial t}+rot\vec{\boldsymbol{E}}\right)\Lambda=0$$

Thus, we find that the replaced expression for L_R is invariant under the transformations (11). Accordingly, by requiring that the Lagrangian $L_N^\circ+L_P^\circ+L_R$ must be invariant under the transformations (11), we can obtain the condition $s=0$, and hence, we can obtain the P- and T-invariant theory for this case. But, these transformations can not

make the Lagrangian $L_N+L_P+L_R$ invariant. To make the part of the Lagrangian L_N+L_P invariant, they must be modified as

$$\left.\begin{aligned} & e^{(\alpha^*-\alpha)\gamma_5}\frac{1}{2}(1+\tau_3)\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu} \\ & \qquad \to e^{(\alpha^*-\alpha)\gamma_5}\frac{1}{2}(1+\tau_3)\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}+ie^{(\alpha+\alpha^*)\gamma_5}\gamma_\mu\partial_\mu\Lambda, \\ & N\to e^{-i(e'/\kappa)\Lambda}N. \end{aligned}\right\} \qquad (13)$$

As for L_R, if we add the further condition to the constant spinor ϵ, that is*

$$\bar{\epsilon}\,\frac{1}{2}(1+\tau_3)\gamma_5\epsilon=0, \qquad (14)$$

we can replace it by

$$\begin{aligned} L_R &= -\frac{1}{4}\int F_{\mu\nu}F_{\mu\nu}d^4x \\ &\to -\bar{\epsilon}\,\frac{1}{8}\,\frac{1}{\cosh 2(\alpha^*-\alpha)}\int e^{2(\alpha^*-\alpha)\gamma_5}\frac{1}{2}(1+\tau_3)(\sigma_{\mu\nu}F_{\mu\nu})^2d^4x\cdot\epsilon \end{aligned}$$

in which, we used the relation $e^{a\gamma_5}=\cosh a+\gamma_5\sinh a$.

The condition (14) can be compatible with the conditions (7). Here, we must pay attention to the fact that, unlike the vectorial case, the transformations (11) and (13) have the different forms unless $\alpha=0$.

Now, it seems to be reasonable to postulate the requirement that the quantum electrodynamics must be independent of the mixing ratio of parity for the fermion in all of its aspects, especially, the extended gauge transformations which make the formalism be invariant when only one interaction among the vectorial and the Pauli-type interactions is present must be the same for any value of the mixing ratio for the fermion. In the following, we will call this requirement the "*equivalence requirement*" and will write it shortly as E. R. Hence, if we require that the interactions which do not satisfy E. R. should be excluded, we can get rid of the Pauli-term from Q. E. D., and thus, we can conclude that the electromagnetic interaction can be obtained from the free Lagrangian by the replacement

$$\partial_\mu \to \partial_\mu - ie\,\frac{1}{2}(1+\tau_3)A_\mu, \qquad (15)$$

which shows the "principle of the minimal electromagnetic interaction"[5]. This selection rule is found to be applicable to the cases of the strong mesonic interactions, as will be stated later.

§. 3 Pseudo Q. E. D.

Let us consider the possibility that the electromagnetic potential is a pseudovector.

* Of course, to remove the term containing γ_5, we may put the second condition on α i.e., $\alpha^*=\alpha$, instead of the condition (14) for ϵ. But, as it is unnecessary to restrict the value of α to be real, we adopted the condition (14). C. f. §.5 Discussions.

We are good enough to consider only the vectorial interaction. Denoting the pseudovector potential as B_μ, then, the interaction Lagrangian is, in general, written as

$$\left.\begin{aligned} L &= L_N + L^P_{int} + L^P_R, \\ L^P_{int} &= -ie\int N^+ e^{(\alpha+\alpha^*)\gamma_5} e^{-\delta\gamma_5}\gamma_5\gamma_\mu B_\mu \frac{1}{2}(1+\tau_3)N d^4x, \\ L^P_R &= -\frac{1}{2}\int\left\{\frac{1}{2}G_{\mu\nu}G_{\mu\nu} - (\partial_\mu B_\mu)^2\right\}d^4x, \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (16)$$

where

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu.$$

The hermiticity condition is found to be

$$\delta = \delta^*.$$

This is the same condition as in Q. E. D. The equations of motion derived from this Lagrangian by variating B_μ independently become

$$\begin{aligned} (\gamma_\mu\partial_\mu + \kappa e^{-2\alpha\gamma_5})N &= -ie\, e^{-\delta\gamma_5}\gamma_5\gamma_\mu B_\mu \frac{1}{2}(1+\tau_3)N, \\ \Box B_\mu &= -j^P_\mu, \end{aligned} \quad \cdots\cdots\cdots\cdots (17)$$

where

$$j^P_\mu = ieN^+ e^{(\alpha+\alpha^*)\gamma_5} e^{-\delta\gamma_5}\gamma_5\gamma_\mu \frac{1}{2}(1+\tau_3)N$$

The quantity j^P_μ is the source of the B_μ field and therefore should be understood as the charge-current density. But, in this case, we can no lomger derive the conservation law for this charge current density. Indeed, if we calculate the expression $\partial_\mu j^P_\mu$ directly from the equation (17), then, as δ is real, we obtain

$$\partial_\mu j^P_\mu = -ie\kappa N^+ e^{(\alpha^*-\alpha)\gamma_5}\gamma_5(e^{\delta\gamma_5} + e^{-\delta\gamma_5})\frac{1}{2}(1+\tau_3)N. \quad \cdots\cdots\cdots\cdots (18)$$

The right side can not vanish for any value of the constant δ. The extended gauge transformations in this case read

$$\begin{aligned} \frac{1}{2}(1+\tau_3)\gamma_5 A_\mu &\to \frac{1}{2}(1+\tau_3)\gamma_5 A_\mu + I\partial_\mu\Lambda, \\ N &\to e^{ie\Lambda}N. \end{aligned}$$

The requirement that the formalism should be extended gauge invariant restrict the value of δ to zero. The corresponding identical relation is, from the expression (18),

$$\partial_\mu j^P_\mu = -2ie\kappa N^+ e^{(\alpha^*-\alpha)\gamma_5}\gamma_5 \frac{1}{2}(1+\tau_3)N \quad \cdots\cdots\cdots\cdots (19)$$

or applying R. T. (9),

$$\partial_\mu j^{0P}_\mu = -2ie\kappa \check{N}^+\gamma_5 \frac{1}{2}(1+\tau_3)\check{N}. \quad \cdots\cdots\cdots\cdots (19')$$

Thus, so long as the charge-current conservation is strongly required for Q. E. D., the possibility of the photon wave function being pseudovector must be excluded.

In the Lagrangian (16), we can not define L_R^P as

$$L_R^P=-\frac{1}{4}\int G_{\mu\nu}G_{\mu\nu}d^4x. \qquad (20)$$

If we choose L_R^P as (20), then variating Lagrangian with B_μ, we get as the equation for B_μ,

$$(\Box\delta_{\mu\nu}-\partial_\mu\partial_\nu)B_\nu=-j_\mu^P.$$

Applying the differential operator ∂_μ on both side of this equation, as the left side vanishes identically, we must have

$$\partial_\mu j_\mu^P=0.$$

This relation contradict with the relation (20). Thus, we find that the pseudo-radiation part of the Lagrangian L_R^P must be so chosen as the expression (16) and not as (20). Besides, we find that the four-divergence of the pseudovector potential B_μ does not vanish, i. e., $\partial_\mu B_\mu\neq 0$.

In the above arguments, we consider the wave function B_μ is entirely independent of the ordinary photon wave function A_μ. We can, however, relate B_μ to A_μ in the following way. Introducing the relative tensor (Levi-Civita density) $\epsilon_{\mu\nu\lambda\rho}$ of weight-1* and define the tensor $G_{\mu\nu}$ by the relations

$$G_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho},\quad F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\lambda\rho}G_{\lambda\rho}, \qquad (21)$$

where

$$G_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu,\quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu.$$

Using the relation

$$\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\kappa\lambda\gamma\delta}=\delta_{\alpha\beta,\kappa\lambda}=\delta_{\alpha\kappa}\delta_{\beta\lambda}-\delta_{\kappa\beta}\delta_{\lambda\alpha},$$

we easily find that

$$G_{\mu\nu}G_{\mu\nu}=F_{\mu\nu}F_{\mu\nu}.$$

From the relations (21), we get

$$G_{\mu\nu}=\epsilon_{\mu\nu\lambda\rho}\frac{\partial A_\rho}{\partial x_\lambda}.$$

Now, let us consider that there is such a interaction term as

$$\frac{1}{4}j_\mu\int^{x_\nu}G_{\mu\nu}dx_\nu \qquad (22)$$

in the Lagrangian density, where j_μ is defined by the expression (4) with $\delta=0$. This

* C. f. E. M. Corson, Introduction to Tensors, Spinors and Relativistic Wave Equations.

interaction does not conserve the parity. Then, the corresponding Lagrangian $\int d^4xj_\mu\cdot\int G_{\mu\nu}dx_\nu$ is calculated as:

$$
\begin{aligned}
&=\frac{1}{4}\left[\int d^4xj_\mu\int\frac{\partial B_\nu}{\partial x_\mu}dx_\nu-\int d^4xj_\mu\int\frac{\partial B_\mu}{\partial x_\nu}dx_\nu\right]\\
&=\frac{1}{4}\left[\int d^4xj_\mu\partial_\mu\int B_\nu dx_\nu-4\int d^4xj_\mu B_\mu\right]\\
&=\frac{1}{4}\left[\int d^4x\partial_\mu[j_\mu\int B_\nu dx_\nu]-\int d^4x\partial_\mu j_\mu\int B_\nu dx_\nu\right]-\int d^4xj_\mu B_\mu .
\end{aligned}
$$

As the first and the second terms of the right side vanish, the interaction term (22) is effectively equal to $-j_\mu B_\mu$. That is to say, in the Lagrangian, we can put as

$$
B_\mu=-\frac{1}{4}\int G_{\mu\nu}dx_\nu=-\frac{1}{4}\epsilon_{\mu\nu\lambda\rho}\int\frac{\partial A_\rho}{\partial x_\lambda}dx_\nu . \qquad (23)
$$

This is the potential used by Sachs[9]. It is remarkable that the potential B_μ defined by (23) satisfies the condition $\partial_\mu B_\mu=0$ and further does not contradict with the requirement of the gauge invariance. The interaction of the type (22), however, meets the grave difficulty concerning to the quantization of the theory, the details of which will be given in the other place.[10]

§ . 4 Strong Mesonic Interactions

1°) Pion-Nucleon System

The interactions between the pion and the nucleon has the form $\overline{N}O(\vec{\tau}\cdot\vec{\pi})N$. The total Lagrangian of the system can be written as

$$
\left.
\begin{aligned}
&L=L_N+L_{N\Pi}+L_\Pi,\quad L_{N\Pi}=L_{ps}+L_{pv},\\
&L_N=-\int\overline{N}(e^{A\gamma_5}\gamma_\mu\partial_\mu+\kappa e^{-B\gamma_5})Nd^4x,\\
&L_{ps}=-ig_{ps}\int\overline{N}e^{\beta\gamma_5}\gamma_5\vec{\tau}\cdot\vec{\pi}Nd^4x,\\
&L_{pv}=-ig_{pv}\int\overline{N}e^{\delta\gamma_5}\gamma_5\gamma_\mu\partial_\mu(\vec{\tau}\cdot\vec{\pi})Nd^4x,\\
&L_\Pi=-\frac{1}{2}\int(\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+m^2\vec{\pi}\cdot\vec{\pi})d^4x,
\end{aligned}
\right\} \qquad (24)
$$

where m stands for the pion mass. As is in Q. E. D., the hermiticity conditions of $L_{N\pi}$ become

$$
\beta=-B+ip,\quad \delta=A+q \qquad (25)
$$

where the constants p and q are real arbitrary. Using these conditions L_{ps} and L_{pv} become

$$\left.\begin{aligned}L_{ps}&=-ig_{ps}\int N^{+}e^{(\alpha+\alpha)\gamma_5}e^{p\gamma_5}\gamma_5\vec{\tau}\cdot\vec{\pi}Nd^4x,\\ L_{pv}&=-ig_{pv}\int N^{+}e^{(\alpha+\alpha)\gamma_5}e^{p\gamma_5}\gamma_5\gamma_\mu\partial_\mu(\vec{\tau}\cdot\vec{\pi})Nd^4x,\\ &\qquad(\alpha=A+Б)\end{aligned}\right\}\quad\cdots\cdots(26)$$

respectively. Transforming the generalized wave function N to the ordinary one $\overset{c}{N}$ by R. T. (9), we get

$$\left.\begin{aligned}L=&-\int\{\overset{c}{N}^{+}(\gamma_\mu\partial_\mu+\kappa)\overset{c}{N}+ig_{ps}\overset{c}{N}^{+}e^{p\gamma_5}\gamma_5\vec{\tau}\cdot\vec{\pi}\overset{c}{N}\\ &+ig_{pv}\overset{c}{N}^{+}e^{p\gamma_5}\gamma_5\gamma_\mu\partial_\mu(\vec{\tau}\cdot\vec{\pi})\overset{c}{N}\}d^4x+L_\pi\end{aligned}\right\}\quad\cdots\cdots(27)$$

This is not P- and T-invariant as it should be. In order to make it P- and T-invariant, we must have $p=q=0$, but usually as there is no such a property for the pionnucleon system as a property of a gauge-invariance for Q. E. D., we can not put the above conditions.

Now, we will follow the case of Q. E. D., and require the following properties to the Lagrangian, that is: the Lagrangian (27) must be invariant under the transformations

$$\left.\begin{aligned}&\gamma_5\vec{\tau}\cdot\vec{\pi}\to\gamma_5\vec{\tau}\cdot\vec{\pi}+\Lambda\cdot I,\\ &\overset{c}{N}\to e^{-ig_{pv}\Lambda}\overset{c}{N},\end{aligned}\right\}\quad\cdots\cdots(28)$$

when $g_{ps}=0$, $g_{pv}\neq0$;

$$\left.\begin{aligned}&\gamma_5\vec{\tau}\cdot\vec{\pi}\to\gamma_5\vec{\tau}\cdot\vec{\pi}+\gamma_\mu\partial_\mu\Lambda,\\ &\overset{c}{N}\to e^{-ig_{ps}\Lambda}\overset{c}{N},\end{aligned}\right\}\quad\cdots\cdots(29)$$

when $g_{ps}\neq0$, $g_{pv}=0$;

where Λ is a C-number space-time scalar function obeying the relation

$$(\Box-m^2)\Lambda=0.$$

As was in Q. E. D., L_π is to be considered, introducing the constant spinor ϵ defined by the relations (7) as

$$\left.\begin{aligned}&\partial_\mu\vec{\pi}\cdot\partial_\mu\vec{\pi}+m^2\vec{\pi}\cdot\vec{\pi}\\ &=\bar{\epsilon}\{\partial_\mu\vec{\tau}\cdot\partial_\mu\vec{\pi}+m^2\vec{\tau}\cdot\vec{\pi}\}\epsilon\\ &=\bar{\epsilon}\{\partial_\mu(\gamma_5\vec{\tau}\cdot\vec{\pi})\partial_\mu(\gamma_5\vec{\tau}\cdot\vec{\pi})+m^2(\gamma_5\vec{\tau}\cdot\vec{\pi})(\gamma_5\vec{\tau}\cdot\vec{\pi})\}\epsilon,\end{aligned}\right\}\quad\cdots\cdots(30)$$

where we used the relations

$$(\vec{\tau}\cdot\vec{A})(\vec{\tau}\cdot\vec{B})=(\vec{A}\cdot\vec{B})+i(\vec{\tau}\cdot[\vec{A}\times\vec{B}])$$

and

$$[\vec{\tau}\cdot\vec{\pi}]=0.$$

The expression (30) is found to be invariant under the transformations (28) and (29). Requiring the above restrictions, we can get $p=0$ or $q=0$ as the same way as in Q. E. D., and are able to obtain the P- and T-invariant theory. In this case, however,

there is no transformations under which both of the direct and the derivative interactions remain invariant. Therefore, if one of them be allowed, then another one should be abandoned.

Now, it is reasonable to impose E. R. to the pion-nucleon system as was done in Q. E. D.. For the pion-nucleon system, E. R. takes the following expression, that is: for both of the common theory using the ordinary Dirac equation and the generalized theory using the generalized Dirac equation, the formalism for the pion-nucleon system should be invariant under the same transformations. The derivative interaction is easily proved to satisfy the above requirement. As for the direct one, the transformations which make the generalized theory invariant must take the form

$$e^{(\alpha^{*}-\alpha)\gamma_5}\gamma_5\vec{\tau}\cdot\vec{\pi}\rightarrow e^{(\alpha^{*}-\alpha)\gamma_5}\vec{\tau}\cdot\vec{\pi}+e^{(\alpha+\alpha^{*})\gamma_5}\gamma_\mu\partial_\mu\Lambda,$$
$$N\rightarrow e^{-ig_{ps}\Lambda}N.$$

Apparently, these have the different form from that of the transformations (22). Hence, the direct interaction should be excluded so long as we postulate E. R., and thus, we can say that the type of the interaction for the pion-nucleon system is derivative.

2°) Pion-Baryon System

As was shown in the above discussions, we find that, for the pion-nucleon system, there is the extended type of the gauge transformations under which the theory is invariant and that the requirement E. R. for the form of that transformations assure the invariances of the theory under the P- and T-transformations. For the cases of the strong interactions between the pion and the baryon, in order to exist such extended gauge transformations, the form of the interaction should be written as

$$\bar{B}\vec{\tau}\cdot\vec{\pi}B, \quad\cdots\cdots\cdots\cdots\quad (31)$$

where B stands for the iso-spinor wave function of the baryon, and we did not write the γ-matrices explicitly. Therefore, whether the system of the pion and the baryon obeys our assumption concerning the strong interactions depends upon whether the form of the interaction of the system has or can have the form (31).

Now, the allowed forms of the interactions between the pion and the baryon are

$$g_1\bar{N}\vec{\tau}\cdot\vec{\pi}N, \quad\cdots\cdots\cdots\cdots\quad (32\text{-}1)$$

$$g_2\{\bar{\Lambda}\vec{\Sigma}\cdot\vec{\pi}+\bar{\vec{\Sigma}}\cdot\vec{\pi}\Lambda\}, \quad\cdots\cdots\cdots\cdots\quad (32\text{-}2)$$

$$g_3 i(\bar{\vec{\Sigma}}\times\vec{\Sigma})\cdot\vec{\pi}, \quad\cdots\cdots\cdots\cdots\quad (32\text{-}3)$$

$$g_4\bar{\Xi}\vec{\tau}\cdot\vec{\pi}\Xi. \quad\cdots\cdots\cdots\cdots\quad (32\text{-}4)$$

The interaction (32-4) has the same form as (32-1) and can be treated equally as (32-1). Therefore, the question reduces to whether we can construct the form (31) from (32-2) and (32-3). The Gell-Mann's assumption of the "global symmetry" shows the very possibility for it. Accordingly, we can conclude that; if the strong mesonic interactions satisfy E. R., they must be written as,

$$g_i \bar{B}_i \vec{\tau}\cdot\vec{\pi} B_i, \quad i=1,2,3,4, \qquad (33)$$

where

$$B_1=N, \quad B_4=\Xi, \qquad (34)$$

$$B_2=\frac{1}{\sqrt{2}}\begin{pmatrix}\Sigma_1-i\Sigma_2\\ \Lambda+\Sigma_3\end{pmatrix}=\begin{pmatrix}\Sigma^+\\ Y\end{pmatrix}, \quad B_3=\frac{1}{\sqrt{2}}\begin{pmatrix}\Lambda-\Sigma_3\\ \Sigma_1+i\Sigma_2\end{pmatrix}=\begin{pmatrix}Z\\ \Sigma^-\end{pmatrix},$$

provided that $g_2=g_3$, or alternatively; if the strong mesonic interactions are globally symmetric, they satisfy E. R. The type of the interaction is, of course, derivative, and the invariances under the P- and T-transformations are thus assured.

3°) Strong Interactions between the Baryons and the Iso-Bose Bosons with Strangeness Zero

The treatment for the iso-vector bosons is almost the same as the one for the pion mentioned above. Let us write the wave function of the iso-vector boson as $\vec{V}$. Then the form of the interaction between $\vec{V}$ and the baryon that satisfy E. R. must be

$$g\bar{B}O\vec{\tau}\cdot\partial\vec{V}B, \qquad (35)$$

where O means the suitable choice of the γ-matrices and ∂ the corresponding differential operator. As for the iso-scalar boson, if we denote the wave function of it as S, then, the results obtained for the iso-vector boson also hold for the iso-scalar boson only if we substitute S in place of $\vec{\tau}\cdot\vec{V}$. Therefore, we will discuss in the following about the iso-vector boson.

Concerning to the properties for E. R., the matrices O are classified into two groups, that is[1),2)]

a) *1st group:* γ_μ, $\gamma_5\gamma_\mu$;

b) *2nd group:* 1, γ_5, $\gamma_\mu\gamma_\nu$, $\gamma_5\gamma_\mu\gamma_\nu$.

The first group satisfy E. R. while the second group does not. Thus, for the strong interactions, the second group must be excluded. In the following, we will examine this for the spin *0* and *1* particles separately.

a°) The First Group

When the ordinary Dirac equations are used, the formalisms become invariant under the following extended gauge transformations:

$$\left.\begin{aligned}\Gamma\gamma(\vec{\tau}\cdot\vec{\pi}) &\to \Gamma\gamma(\vec{\tau}\cdot\vec{\pi}) - \Gamma_i L,\\ \vec{B} &\to e^{-ig\Lambda}\vec{B},\end{aligned}\right\} \qquad (36)$$

where

$$\Gamma=\begin{cases}1 & \textit{for even parity } \vec{V},\\ \gamma_5 & \textit{for odd parity } \vec{V};\end{cases}$$

$$\gamma=\begin{cases}1 & \textit{for } \vec{V} \textit{ of spin } 0,\\ \gamma_\mu & \textit{for } \vec{V} \textit{ of spin } 1;\end{cases}$$

$$L=\begin{cases}\Lambda & for\ \vec{V}\ of\ spin\ 0,\\ \partial_\mu\Lambda & for\ \vec{V}\ of\ spin\ 1;\end{cases}$$

and the scalar function Λ satisfy the relation

$$(\square-M^2)\Lambda=0,$$

where M stands for the mass of the iso-vector boson. As is easily shown, if we replace $\overset{c}{B}$ by B, these transformations also make the theory in which the generalized Dirac equations are used instead of the ordinary ones invariant. The interactions belong to this group are the derivative coupling of the spin O boson and the direct coupling of the spin 1 particle. These interactions are all assured to be P– and T–invariant.

b°) The Second Group

(i). Spin 0 Boson with Direct Coupling

For the ordinary theory, the formalisms are invariant under the extended gauge transformations

$$\left.\begin{aligned}\Gamma\vec{\tau}\cdot\vec{V}&\to\Gamma\vec{\tau}\cdot\vec{V}+\gamma_\mu\partial_\mu\Lambda,\\ \overset{c}{B}&\to e^{-ig\Lambda}\overset{c}{B}.\end{aligned}\right\}\qquad\cdots\cdots(37)$$

As is stated previously for the pion-nucleon system, the forms of the transformations which make the generalized theory invariant are different from the above, and so, the direct couplings for the spin 0 bosons should be excluded.

(ii). Spin 1 Boson with Derivative Coupling

In this case, the difference appears between the iso-scalar boson and the iso-vector boson. For the iso-vector Boson, the hermitic total Lagrangian is, when the ordinary Dirac equation is used, written as

$$\left.\begin{aligned}L&=L_B+L_{int}+L_v,\\ L_B^c&=-\int\overset{c}{B}^+(\gamma_\mu\partial_\mu+\kappa)\overset{c}{B}d^4x,\\ L_{int}^c&=-ig\int\overset{c}{B}^+e^{it\gamma_5}\Gamma\gamma_\mu\gamma_\nu\partial_\mu(\vec{\tau}\cdot\vec{V}_\nu)\overset{c}{B}d^4x,\\ L_v&=-\frac{1}{2}\int\left\{(\partial_\mu\vec{V}_\nu)(\partial_\mu\vec{V}_\nu)+M^2\vec{V}_\nu\cdot\vec{V}_\nu\right\}d^4x,\end{aligned}\right\}\qquad\cdots\cdots(38)$$

and, when the generalized Dirac equation is used, as

$$\left.\begin{aligned}L&=L_B+L_{int}+L_v,\\ L_B&=-\int\overline{B}(e^{A\gamma_5}\gamma_\mu\partial_\mu+\kappa e^{-B\gamma_5})Bd^4x,\\ L_{int}&=-ig\int B^+e^{(\alpha^*-\alpha)\gamma_5}e^{it\gamma_5}\Gamma\gamma_\mu\gamma_\nu\partial_\mu(\vec{\tau}\cdot\vec{V}_\nu)Bd^4x,\end{aligned}\right\}\qquad\cdots\cdots(39)$$

where the wave function $\vec{V}$ satisfies the relation

$$\partial_\mu \vec{V}_\mu = 0. \tag{40}$$

When $t=0$, the part of the Lagrangian $L_B' + L_{int}'$ is found to be invariant under the transformations

$$\left.\begin{aligned} &\Gamma\gamma_\mu\gamma_\nu\partial_\mu(\vec{\tau}\cdot\vec{V}_\nu) \rightarrow \Gamma\gamma_\mu\gamma_\nu\partial_\mu(\vec{\tau}\cdot\vec{V}_\nu) + \gamma_\mu\partial_\mu\Lambda, \\ &\overset{c}{B} \rightarrow e^{-ig\Lambda}\overset{c}{B}, \quad (\Box - M^2)\Lambda = 0. \end{aligned}\right\} \tag{41}$$

Using the relation (40), these can also be written as

$$\left.\begin{aligned} &\Gamma\gamma_\mu(\vec{\tau}\vec{V}_\mu) \rightarrow \Gamma\gamma_\mu(\vec{\tau}\vec{V}_\mu) + \Lambda \\ &\overset{c}{B} \rightarrow e^{-ig\Lambda}\overset{c}{B}, \quad (\Box - M^2)\Lambda = 0, \end{aligned}\right\} \tag{42}$$

where the positive sign corresponds to $\Gamma=1$ and the negative sign to $\Gamma=\gamma_5$. Next, by introducing the constant spinor ϵ defined by the relations (7), we examine whether L_ν is invariant under the above transformations (42). We have

$$\begin{aligned} &\Gamma\gamma_\mu(\vec{\tau}\cdot\vec{V}_\mu)\Gamma\gamma_\nu(\vec{\tau}\cdot\vec{V}_\nu) \\ &= \pm\frac{1}{2}\{(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)(\vec{\tau}\cdot\vec{V}_\mu)(\vec{\tau}\cdot\vec{V}_\nu) \\ &\qquad\qquad \mp\gamma_\nu\gamma_\mu[(\vec{\tau}\cdot\vec{V}_\nu)(\vec{\tau}\cdot\vec{V}_\mu) - (\vec{\tau}\cdot\vec{V}_\mu)(\vec{\tau}\cdot\vec{V}_\nu)]\} \\ &= \pm\vec{V}_\mu\cdot\vec{V}_\mu \pm i\frac{1}{2}\gamma_\nu\gamma_\mu\vec{\tau}\cdot\{[\vec{V}_\nu\times\vec{V}_\mu] - [\vec{V}_\mu\times\vec{V}_\nu]\} \\ &= \pm\vec{V}_\mu\cdot\vec{V}_\mu \pm i\gamma_\mu\gamma_\nu\vec{\tau}\cdot[\vec{V}_\mu\times\vec{V}_\nu], \end{aligned}$$

and

$$\begin{aligned} &\Gamma\gamma_\mu\partial_\rho(\vec{\tau}\cdot\vec{V}_\mu)\cdot\Gamma\gamma_\nu\partial_\rho(\vec{\tau}\cdot\vec{V}_\nu) \\ &= \pm\frac{1}{2}\{\gamma_\mu\gamma_\nu(\vec{\tau}\cdot\partial_\rho\vec{V}_\mu)(\vec{\tau}\cdot\partial_\rho\vec{V}_\nu) + \gamma_\nu\gamma_\mu(\vec{\tau}\cdot\partial_\rho\vec{V}_\nu)(\vec{\tau}\cdot\partial_\rho\vec{V}_\mu)\} \\ &= \pm(\partial_\nu\vec{V}_\mu)(\partial_\nu\vec{V}_\mu) \pm i\gamma_\mu\gamma_\nu\vec{\tau}\cdot[\partial_\rho\vec{V}_\mu\times\partial_\rho\vec{V}_\nu] \end{aligned}$$

Therefore,

$$\begin{aligned} &\int\{\Gamma\gamma_\mu\partial_\rho(\vec{\tau}\cdot\vec{V}_\mu)\cdot\Gamma\gamma_\nu\partial_\rho(\vec{\tau}\cdot\vec{V}_\nu) + M^2\Gamma\gamma_\mu(\vec{\tau}\cdot\vec{V}_\mu)\cdot\Gamma\gamma_\nu(\vec{\tau}\cdot\vec{V}_\nu)\}d^4x \\ &= \pm\int\{(\partial_\nu\vec{V}_\mu)(\partial_\nu\vec{V}_\mu) + M^2\vec{V}_\mu\cdot\vec{V}_\mu\}d^4x\cdot I \\ &\qquad \pm i\gamma_\mu\gamma_\nu\vec{\tau}\cdot\int\{[\partial_\rho\vec{V}_\mu\times\partial_\rho\vec{V}_\nu] + M^2[\vec{V}_\mu\times\vec{V}_\nu]\}d^4x. \end{aligned}$$

For the iso-vector boson, the second term of the right side vanishes identically, and so, the free iso-scalar boson part of the Lagrangian L_s can be replaced as

$$L_s \rightarrow -\frac{1}{2}(\pm)\bar{\epsilon}\int\{\Gamma\gamma_\mu\partial_\rho S_\mu\cdot\Gamma\gamma_\nu\partial_\rho S_\nu + M^2\Gamma\gamma_\mu S_\mu\cdot\Gamma\gamma_\nu S_\nu\}\epsilon\, d^4x.$$

But, for the iso-vector boson, as it does not vanish identically, in general, we can not make such a replacement as above unless we put the further conditions on the constant spinor ϵ. This can be done by taking the trace of the expression instead of introducing the constant spinor ϵ. Of course, taking the trace is equivalent to choice the constant

spinor ϵ specially. Thus, for the iso-vector boson, we replace the free part of the Lagrangian L_ν as

$$L_\nu\to \frac{1}{2}(\pm)\frac{1}{8}\,Trace\int\{\Gamma\gamma_\mu\partial_\rho(\vec{\tau}\cdot\vec{\boldsymbol{V}}_\mu)\cdot\Gamma\gamma_\nu\partial_\rho(\vec{\tau}\cdot\vec{\boldsymbol{V}}_\nu)$$
$$+M^2\Gamma\gamma_\mu(\vec{\tau}\cdot\vec{\boldsymbol{V}}_\mu)\cdot\Gamma\gamma_\nu(\vec{\tau}\cdot\vec{\boldsymbol{V}}_\nu)\}d^4x.$$

If we apply the transformations (42) to the latter expressions, we get

$$L_\nu\to L_\nu-\frac{1}{2}(\pm)\frac{1}{8}\,Trace\int(\partial_\rho\Lambda\cdot\partial_\rho\Lambda+M^2\Lambda^2)d^4x$$
$$-\frac{1}{2}(\pm)\frac{1}{8}\,Trace\int\Big\{\Gamma\gamma_\mu\partial_\rho(\vec{\tau}\cdot\vec{\boldsymbol{V}}_\mu)\cdot\partial_\rho\Lambda+\partial_\rho\Lambda\cdot\Gamma\gamma_\mu\partial_\rho(\vec{\tau}\cdot\vec{\boldsymbol{V}}_\mu)$$
$$+M^2\Big(\Gamma\gamma_\mu(\vec{\tau}\cdot\vec{\boldsymbol{V}}_\mu)\Lambda+\Lambda\Gamma\gamma_\mu(\vec{\tau}\cdot\vec{\boldsymbol{V}}_\mu)\Big)\Big\}d^4x$$

The second term of the right side vanishes due to the condition $(\Box-M^2)\Lambda=0$. The third term can be rewritten as

$$\sim\int\Gamma\gamma_\mu(\vec{\tau}\cdot\vec{\boldsymbol{V}}_\mu)(\Box-M^2)\Lambda d^4x$$

and is found to be zero. Therefore, we find that is invariant under the transformations (42).

Now, if we start from the Lagrangian (39), then the corresponding transformations will have the form

$$e^{(\alpha^*-\alpha)\gamma_5}\Gamma\gamma_\mu(\vec{\tau}\vec{\boldsymbol{V}}_\mu)\to e^{(\alpha^*-\alpha)\gamma_5}\Gamma\gamma_\mu(\vec{\tau}\vec{\boldsymbol{V}}_\mu)\pm e^{(\alpha+\alpha^*)\gamma_5}\Lambda,$$
$$B\to e^{-ig\Lambda}B,\quad(\Box-M^2)\Lambda=0,$$

which are different from (42). Therefore, so long as we require E. R., the derivative coupling of the spin 1 particle should be excluded.

Putting all accounts together, we can conclude that; if there exist the iso-bose bosons with spin 0 and with strangeness 0 in the nature, they must be pseudoscalar particles with the pseudovector couplings, and that; the form of the strong interaction of the iso-bose boson with spin 1 and with strangeness 0 must be direct. There is no other particles with spin 0 or 1 in the nature.

In our case of the strong interactions between the baryons and the iso-bose boson with strangeness zero, according to the existence of the proposed extended gauge invariance, it is found that the interaction terms can be obtained from the free Lagrangian for the baryon by the substitutions

$$\gamma_\mu\partial_\mu\to\gamma_\mu\partial_\mu+ig\Gamma\gamma_\mu\vec{\tau}\cdot\vec{\boldsymbol{V}}$$

for the iso-vector boson, and

$$\gamma_\mu\partial_\mu\to\gamma_\mu\partial_\mu+ig\Gamma\gamma_\mu S$$

for the iso-scalar boson. This fact may be called as the "*principle of the minimal mesonic interaction*" corresponding to the "principle of the minimal electromagnetic interaction".

Finally, we will list up the relations corresponding to the invariances under the extended gauge transformations in the cases of the iso-vector and the iso-scalar bosons

with spin 0 (of course, derivative coupling).

1) *iso-scalar scalar*

$$g_v\partial_\mu(B^+e^{(\alpha+\alpha^*)\gamma_5}\gamma_\mu B)=0,$$

$$or\quad g_v\partial_\mu(\overset{c}{B}{}^+\gamma_\mu\overset{c}{B})=0;$$

2) *iso-scalar pseudoscalar*

$$g_{pv}\partial_\mu(B^+e^{(\alpha+\alpha^*)\gamma_5}\gamma_\mu\gamma_5 B)$$
$$=2\kappa g_{pv}B^+e^{(\alpha^*-\alpha)\gamma_5}\gamma_5 B,$$

$$or\quad g_{pv}\partial_\mu(\overset{c}{B}{}^+\gamma_\mu\gamma_5\overset{c}{B})=2\kappa g_{pv}\overset{c}{B}{}^+\gamma_5\overset{c}{B};$$

3) *iso-vector scalar*

$$g_v\partial_\mu(B^+e^{(\alpha+\alpha^*)\gamma_5}\gamma_\mu\vec{\tau}B)$$
$$=-2g_v^2B^+e^{(\alpha+\alpha^*)\gamma_5}\gamma_\mu\partial_\mu[\vec{\tau}\times\vec{V}]B,$$

$$or\quad g_v\partial_\mu(\overset{c}{B}{}^+\gamma_\mu\vec{\tau}\overset{c}{B})=-2g_v^2\overset{c}{B}{}^+\gamma_\mu\partial_\mu[\vec{\tau}\times\vec{V}]\overset{c}{B};$$

4) *iso-vector pseudoscalar*

$$g_{pv}\partial_\mu(B^+e^{(\alpha+\alpha^*)\gamma_5}\gamma_\mu\gamma_5\vec{\tau}B)$$
$$=2\kappa g_{pv}B^+e^{(\alpha^*-\alpha)\gamma_5}\gamma_5\vec{\tau}B$$
$$+2g_{pv}^2B^+e^{(\alpha+\alpha^*)\gamma_5}\gamma_\mu\partial_\mu[\vec{\tau}\times\vec{V}]B,$$

$$or\quad g_{pv}\partial_\mu(\overset{c}{B}{}^+\gamma_\mu\gamma_5\vec{\tau}\overset{c}{B})$$
$$=2\kappa g_{pv}\overset{c}{B}{}^+\gamma_5\vec{\tau}\overset{c}{B}+2g_{pv}^2\overset{c}{B}{}^+\gamma_\mu\partial_\mu[\vec{\tau}\times\vec{V}]B.$$

The left sides of the above formulas are the sources of the iso-bose bosons with the derivative couplings. We easily find that if we put

$$2\kappa g_{pv}=g_{ps},$$

then, the above relations correspond, in the first order approximations, the identity relations between the pseudoscalar and its derivative couplings.

§ . 5 Discussions

In this paper, we consider the gauge invariance for Q. E. D. as only the mathematical freedom of the theory leaving out the puzzling discussions concerning to the nature and the character about it. The requirement of the gauge invariance is so strong that the form of the interaction in Q. E. D. is completely determined even though the invariance of the theory only under the proper Lorentz transformation is required (except for the Pauli-term), and that the invariances of the theory under the P- and T-transformations are also assured[2),11),12)]. Following Q. E. D., after extending the concept of the gauge invariance formally, and requiring that the strong mesonic interactions must be invariant under the extended gauge transformations, we can restrict the strong mesonic interactions to be P- and T-invariant and further we can restrict them to the derivative coupling. Here, we must notice that the points of similarity between the gauge transformations and the extended gauge transformations are only in form. The property of the invariance

under the new transformations must be interpreted as follows: in the strong interactions containing the iso-bose boson with strangeness zero, the expressions

$$\Gamma\gamma(\vec{\tau}\cdot\vec{V}) \text{ or } \Gamma\gamma S$$

are indefinite by the space-time scalar quantity. Of course, the ordinary gauge transformations in Q. E. D. can also be included into this interpretation. We must introduce the constant spinor ϵ for treating the free boson part of the Lagrangian, but unfortunately the physical meaning of it is not yet clear.

Lately, S. Ozaki[4] proposed a new selection rule. He firstly defined the similarity transformation by the relation

$$e^{a\gamma_5}0 \ e^{a\gamma_5}=0',$$

and assumed that only such interactions as the ones keeping their forms invariant under the above defined similarity transformations are allowed. Therefore, according his selection rule, the only possible forms of the interactions are vector and pseudovector both for the strong interactions and for the weak interactions. Here, we have treated only the cases of the strong interactions, and so long as these strong interactions, our results are found to be compatible with his results. If we think about the connection between the similarity transformations defined S. Ozaki and the returning transformation of ours defined by the formula (7) we will easily find out the correspondence of the two selection rules. For getting the selection rule for the strong interactions, we further postulate the equivalence requirement following Q. E. D. This latter requirement does gurantee the invariances of the theory under the P- and T-transformations.

As for the cases of the K-particle, according to the conservation of the strangeness, the strong K interactions of the Yukawa type contain necessarily three different particles and so we can not treat them with the procedure mentioned above for the bosons with strangeness zero. If we assume that the formalism must go back to the ordinary one by applying R. T. for each of the fermions participating the interaction, then, so long as the K-particle has the pure parity, the mixing ratio of the parity of the fermions must be equal for all of the baryons. However, if such an assumption as a compound model is correct for the K-particle, it seems to be unnecessary to consider that the K-particle has the pure parity, and there may happen that the requirement itself that the theory must go back to the ordinary one by R. T. has to be reconsidered[13]. Anyway, it seems to be difficult to find the solution in the cases of the K-particle.

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Appendix

Résumé of the Dirac Matrices

1°) Pauli's Spin Matrices (2 columns 2 rows)

$$\overset{\circ}{\sigma}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \overset{\circ}{\sigma}_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \overset{\circ}{\sigma}_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad I^{\circ}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

$$\overset{\circ}{\sigma}_1{}^2=\overset{\circ}{\sigma}_2{}^2=\overset{\circ}{\sigma}_3{}^2=1,$$
$$\overset{\circ}{\sigma}_1\overset{\circ}{\sigma}_2=-\overset{\circ}{\sigma}_2\overset{\circ}{\sigma}_1=i\overset{\circ}{\sigma}_3,$$
$$\overset{\circ}{\sigma}_2\overset{\circ}{\sigma}_3=-\overset{\circ}{\sigma}_3\overset{\circ}{\sigma}_2=i\overset{\circ}{\sigma}_1,$$
$$\overset{\circ}{\sigma}_3\overset{\circ}{\sigma}_1=-\overset{\circ}{\sigma}_1\overset{\circ}{\sigma}_3=i\overset{\circ}{\sigma}_2,$$

2°) Dirac's Matrices (4 columns 4 rows)

$$\sigma_k=\begin{pmatrix}\overset{\circ}{\sigma}_k & 0\\ 0 & \overset{\circ}{\sigma}_k\end{pmatrix},\quad (k=1,2,3),\quad I=\begin{pmatrix}I^{\circ} & 0\\ 0 & I^{\circ}\end{pmatrix},$$

$$\rho_1=\begin{pmatrix}0 & I^{\circ}\\ I^{\circ} & 0\end{pmatrix},\quad \rho_2=\begin{pmatrix}0 & -iI^{\circ}\\ iI^{\circ} & 0\end{pmatrix},\quad \rho_3=\begin{pmatrix}I^{\circ} & 0\\ 0 & -I^{\circ}\end{pmatrix},$$

$$[\sigma_i,\ \rho_k]_-=0$$

$$\sigma_1^2=\sigma_2^2=\sigma_3^2=1,\qquad \rho_1^2=\rho_2^2=\rho_3^2=1,$$
$$\sigma_2\sigma_3=-\sigma_3\sigma_2=i\sigma_1,\qquad \rho_2\rho_3=-\rho_3\rho_2=i\rho_1,$$
$$\sigma_3\sigma_1=-\sigma_1\sigma_3=i\sigma_2,\qquad \rho_3\rho_1=-\rho_1\rho_3=i\rho_2,$$
$$\sigma_1\sigma_2=-\sigma_2\sigma_1=i\sigma_3,\qquad \rho_1\rho_2=-\rho_2\rho_1=i\rho_3.$$

$$\alpha_k=\rho_1\sigma_k=\begin{pmatrix}0 & \sigma_k\\ \sigma_k & 0\end{pmatrix},\quad \beta=\rho_3$$

$$\sigma_1=-i\alpha_2\alpha_3,\quad \sigma_2=-i\alpha_3\alpha_1,\quad \sigma_3=-i\alpha_1\alpha_2$$

$$\alpha_k\alpha_l+\alpha_l\alpha_k=2\delta_{kl}\cdot I$$
$$\alpha_k\beta+\beta\alpha_k=0$$

$$\alpha_1\sigma_1=\sigma_1\alpha_1=\alpha_2\sigma_2=\sigma_2\alpha_2=\alpha_3\sigma_3=\sigma_3\alpha_3=\rho_1=-\gamma_5$$
$$\alpha_1\sigma_2=\sigma_1\alpha_2=-\sigma_2\alpha_1=-\alpha_2\sigma_1=i\alpha_3$$
$$\alpha_3\sigma_1=\sigma_3\alpha_1=-\sigma_1\alpha_3=-\alpha_1\sigma_3=i\alpha_2$$
$$\alpha_2\sigma_3=\sigma_2\alpha_3=-\sigma_3\alpha_2=-\alpha_3\sigma_2=i\alpha_1$$

$$\sigma_i\alpha_k+\alpha_k\sigma_i=-2\gamma_5\delta_{ik}$$

3°) γ-Matrices

$$\gamma_k=-i\beta\alpha_k=-i\begin{pmatrix}0 & \sigma_k\\ -\sigma_k & 0\end{pmatrix},\quad (k=1,2,3)$$

$$\gamma_4=\beta=\rho_3,\quad \gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4=-\rho_1$$

$$\gamma_k^2=1,\ (k=1,2,3),\quad \gamma_4^2=1,\quad \gamma_5^2=1,$$

$$\gamma_k^+=\gamma_k,\quad \gamma_4^+=\gamma_4,\quad \gamma_5^+=\gamma_5,$$

(γ_k^+ means the hermitian conjugate of γ_k),

$$\alpha_k=i\gamma_4\gamma_k\ (k=1,2,3),$$
$$\beta=\gamma_4,$$
$$\rho_1=-\gamma_5,$$
$$\rho_2=-i\gamma_1\gamma_2\gamma_3,$$
$$\sigma_1=-i\gamma_2\gamma_3=\sigma_{23},$$
$$\sigma_2=-i\gamma_3\gamma_1=\sigma_{31},$$
$$\sigma_3=-i\gamma_1\gamma_2=\sigma_{12},$$
$$\sigma_{\mu\nu}=\frac{1}{2i}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu),$$

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}\cdot I$$
$$\gamma_5\gamma_\mu+\gamma_\mu\gamma_5=0$$

4°) $\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}=(\vec{\sigma}\cdot\vec{H})-i(\vec{\alpha}\cdot\vec{E})$

$$(\vec{\alpha}\cdot\vec{A})(\vec{\alpha}\cdot\vec{B})=(\vec{A}\cdot\vec{B})+i\vec{\sigma}\cdot[\vec{A}\times\vec{B}]$$
$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B})=(\vec{A}\cdot\vec{B})+i\vec{\sigma}\cdot[\vec{A}\times\vec{B}]$$

CERENKOV RADIATION BY THE STRAIGHT LINE CHARGE IN THE UNIFORM CIRCULAR MOTION

Tadashi WATANABE

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Abstract

In this paper the author calculates the čerenkov radiation in the case that the infinitely long straight line charge moves circularly in the medium with a constant speed.

§ 1 Introduction

When the electric charge is moving uniformly in a medium with the larger velocity than the propagation velocity of the light in this medium, the charge emits the electromagnetic radiation. Such phenomenon is called čerenkov radiation and it was researched in detail by Čerenkov.

Theoretically Frank and Tamm explained this phenomenon successfully in the frame of the classical electrodynamics. Afterwards various authors researched it experimentally and theoretically.

Recently some authors attempt to apply this phenomenon to the generation of the electromagnetic waves of the very short wave length.

Up to the present, these authors discussed the case of the electric charge in the uniform motion, while the present author discussed the case of the electric charge moving circularly with a constant speed.

In order to reduce this problem to the simpler two dimensional problem, the author discusses the čerenkov radiation of the infinitely long straight line charge which is moving circularly in the plane perpendicular to it with a constant speed.

§ 2 Calculations of the field quantities*)

In this section, the author calculates the field quantities radiated by the line charge moving circularly with a constant speed. He considers only the a-c parts of the radiated fields and excludes d-c parts.

Notations;

the linear charge density of the line charge	q
the angular speed of the line charge	Ω
the orbit radius of the line charge	a
the dielectric constant of the medium	ε
the magnetic permeability of the medium	μ

* The MKS unit system is used in this paper.

Using the above notations, the charge density ρ and the current density $\boldsymbol{J}$ are expressed by the following formulas in the cylindrical coordinates. (Fig. 1)

$$\left.\begin{aligned}\rho&=q\frac{\delta(r-a)}{r}\,\delta(\theta-\Omega t,\ mod 2\pi),\\ \boldsymbol{J}&=q\Omega a\tau\frac{\delta(r-a)}{r}\,\delta(\theta-\Omega t,\ mod 2\pi).\end{aligned}\right\}\quad\cdots\cdots\ (1)$$

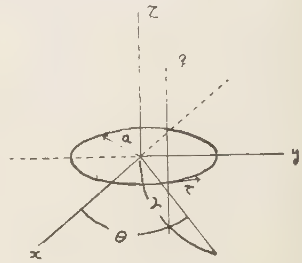

Fig. 1

where τ is the unit vector in the direction of the motion and

$$\delta(\theta-\Omega t,\ mod 2\pi)$$

is defined as follows;

$$\delta(\theta-\Omega t,\ mod 2\pi)=\sum_{n=-\infty}^{\infty}\delta(\theta-\Omega t+2n\pi)\quad\cdots\cdots\ (2)$$

or

$$\delta(\theta-\Omega t,\ mod 2\pi)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{in(\Omega t-\theta)}\quad\cdots\cdots\ (2)'$$

Now he assumes that the medium is isotropic, homogeneous, non-dissipative, and non-dispersive. By (1) and (2)′, the vector potential $\boldsymbol{A}$ and the scalar potential Φ are obtained as follows;

$$\left.\begin{aligned}A_r&=\frac{i\mu qa}{4\pi}\sum_{n=-\infty}^{\infty}{}'\{\psi_{n-1}(n)-\psi_{n+1}(n)\}e^{in(\Omega t-\theta)},\\ A_\theta&=\frac{\mu qa}{4\pi}\sum_{n=-\infty}^{\infty}{}'\{\psi_{n-1}(n)+\psi_{n+1}(n)\}e^{in(\Omega t-\theta)},\\ \Phi&=\frac{q}{2\pi\varepsilon\Omega}\sum_{n=-\infty}^{\infty}{}'\psi_n(n)e^{in(\Omega t-\theta)},\end{aligned}\right\}\quad\cdots\cdots\ (3)$$

where $k=\Omega\sqrt{\varepsilon\mu}$ and $\psi_n(m)=\frac{\pi}{2i}H_n^{(2)}(mkr_>)J_n(mkr_<)$.

$r_>$ is the larger of r and a, and $r_<$ is the smaller of them.

The electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ are derived by (3).

§ 3 Wave-zone approximation

In the case of $kr>ka\gg 1$, the author calculates the wave-zone fields approximately by the expressions (3).

Using the following expansion formulae for the Hankel functions of the positive order;

$$\left.\begin{aligned}H_n^{(1)}(n\ sec\alpha)&\sim\sqrt{\frac{2}{\pi n tg\alpha}}\left\{e^{in(tg\alpha-\alpha)-\frac{\pi}{4}i}+\cdots\cdots\right\},\\ H_n^{(2)}(n\ sec\alpha)&\sim\sqrt{\frac{2}{\pi n tg\alpha}}\left\{e^{-in(tg\alpha-\alpha)+\frac{\pi}{4}i}+\cdots\cdots\right\},\end{aligned}\right\}\quad\cdots\cdots\ (4)$$

The prime of Σ' means that the term $n=0$ is excluded.

the wave-zone parts of $\boldsymbol{E}$ and $\boldsymbol{H}$ are calculated as follows;

$$\left.\begin{aligned}
E_\theta \sim & -\frac{q\mu\Omega a}{4\pi\sqrt{tg\alpha tg\alpha_0}}\left[\sum_{n=1}^{\infty} e^{in(\Omega t-\theta)}\left\{\cos(\alpha-\alpha_0)e^{-inu^{(-)}(\alpha,\alpha_0)}\right.\right.\\
&\left.\left.+i\cos(\alpha+\alpha_0)e^{-inu^{(+)}(\alpha,\alpha_0)}\right\}\right]+\text{complex conjugate},\\
H_z \sim & -\frac{qk a\sin\alpha}{4\pi\sqrt{tg\alpha tg\alpha_0}}\left[\sum_{n=1}^{\infty} e^{in(\Omega t-\theta)}\left\{\cos(\alpha-\alpha_0)e^{-inu^{(-)}(\alpha,\alpha_0)}\right.\right.\\
&\left.\left.+i\cos(\alpha+\alpha_0)e^{-inu^{(+)}(\alpha,\alpha_0)}\right\}\right]+\text{complex conjugate},
\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots (5)$$

where

$$\begin{aligned}
&\sec\alpha=\Omega r,\ \sec\alpha_0=\Omega a,\\
&u^{(+)}(\alpha,\alpha_0)=(tg\alpha-\alpha)+(tg\alpha_0-\alpha_0),\\
&u^{(-)}(\alpha,\alpha_0)=(tg\alpha-\alpha)-(tg\alpha_0-\alpha_0).
\end{aligned}$$

From (5) the following expressions are obtained;

$$\left.\begin{aligned}
E_\theta^{(1)} &\sim -\frac{q\mu\Omega a\cos(\alpha-\alpha_0)}{2\sqrt{tg\alpha tg\alpha_0}}\,\delta(\Omega t-\theta-u^{(-)}(\alpha,\alpha_0);\ mod 2\pi),\\
E_\theta^{(2)} &\sim \frac{q\mu\Omega a\cos(\alpha+\alpha_0)}{2\sqrt{tg\alpha tg\alpha_0}}\,P\frac{1}{2\pi}\cot\frac{1}{2}\left(\Omega t-\theta-u^{(+)}(\alpha,\alpha_0)\right),\\
H_z^{(1)} &\sim -\frac{qk a\sin\alpha}{2\sqrt{tg\alpha tg\alpha_0}}\,\delta(\Omega t-\theta-u^{(-)}(\alpha,\alpha_0),\ mod 2\pi),\\
H_z^{(2)} &\sim \frac{qk a\sin\alpha}{2\sqrt{tg\alpha tg\alpha_0}}\,P\frac{1}{2\pi}\cot\frac{1}{2}\left(\Omega t-\theta-u^{(+)}(\alpha,\alpha_0)\right),\\
E_\theta &= E_\theta^{(1)}+E_\theta^{(2)},\quad H_z=H_z^{(1)}+H_z^{(2)}.
\end{aligned}\right\} \quad \cdots\cdots (6)^*$$

§ 4 The case of the straight line charge in the uniform motion

The author considers the čerenkov radiation of the straight line charge, which is moving uniformly in the plane perpendicular to it with constant velocity $v\mathbf{k}$. $\mathbf{k}$ is the unit vector in the direction of motion of the charge and $v>1/\sqrt{\varepsilon\mu}$.

He takes the cartesian coordinates system so that the line charge is moving in the direction of the z-axix and the line charge is parallel to the x-axix (Fig. 2). And then the expressions for the electric and magnetic fields are obtained as follows;

(i) region $y>0$

*) The symbol P should be refered to W. Heitler, "Quantum theory of radiation" 3rd edition, 1954 p. 69.

$$\left.\begin{aligned} \boldsymbol{E} &= \frac{q}{2\varepsilon}\left\{\frac{\boldsymbol{y}_0}{v} - \nu\boldsymbol{z}_0\right\}\delta\left(t-\nu y-\frac{z}{v}\right), \\ \boldsymbol{H} &= -\frac{q}{2}\,\boldsymbol{x}_0\delta\left(t-\nu y-\frac{z}{v}\right), \end{aligned}\right\} \qquad \cdots\cdots (7)$$

(ii) region $y<0$

$$\left.\begin{aligned} \boldsymbol{E} &= \frac{q}{2\varepsilon}\left\{-\frac{\boldsymbol{y}_0}{v} - \nu\boldsymbol{z}_0\right\}\delta\left(t+\nu y-\frac{z}{v}\right), \\ \boldsymbol{H} &= \frac{q}{2}\,\boldsymbol{x}_0\delta\left(t+\nu y-\frac{z}{v}\right). \end{aligned}\right\} \qquad \cdots\cdots (8)$$

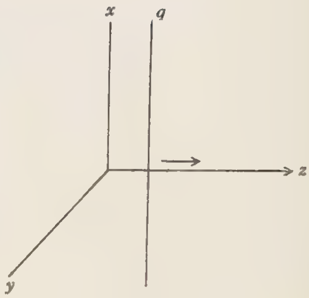

Fig. 2

where $\boldsymbol{x}_0$, $\boldsymbol{y}_0$, and $\boldsymbol{z}_0$ are the unit vectors in the directions of the x-, y-, and z-axis respectively and then $\nu=\sqrt{\varepsilon\mu-\frac{1}{v^2}}$.

§ 5 Discussions

The author's treatment corresponds to the steady state treatment. He thinks that the fields $E_\theta^{(2)}$ and $H_z^{(2)}$ should be called the residual fields and that the appearance of them would be explained by the transient state treatment.

It can easily be shown that in the region $r<a$, the dominant part of radiative power vanishes in the author's steady state considerations.

In the present case, the dominant part of the frequency spectrum of the radiative power per unit interaction length is

$$\frac{q^2}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(1-\frac{1}{(\Omega a)^2\varepsilon\mu}\right)^{\frac{1}{2}} d\omega \; (\omega;\ 0\rightarrow+\infty),$$

where ω is an angular frequency, $ka\gg1$, and $kr\gg1$.

In the case of § 4, the frequency spectrum of the radiative power per unit interaction length is

$$\frac{q^2}{2}\sqrt{\frac{\varepsilon}{\mu}}\left(1-\frac{1}{v^2\varepsilon\mu}\right)^{\frac{1}{2}} d\omega \; (\omega;\ 0\rightarrow+\infty).$$

Finally the author notices that in the case of the point charge in the uniform motion, the spectrum of the radiative power per unit interaction length is

$$\frac{q^2}{2}\mu\left(1-\frac{1}{v^2\varepsilon\mu}\right)\omega d\omega \; (\omega;\ 0\rightarrow+\infty)$$

Note added in proof

After the manuscript was received, the author knew the Dr. Kitao's work*) which contains the three dimensional treatment of this problem.
The author thinks that Dr. Kitao interests in the radiated energy but the present author interests in the field patterns.

*) Kasuo Kitao Prog. Theor. Phys. 23 (1960), 759

NOTE ON CORRECTIONS

Hitosi HIRAMATU

(Received Dec. 1, 1959)

The author published recently his two papers "On Riemannian spaces admitting groups of conformal transformations" (Jour. Math. Soc. of Japan, 9 (1957), 114–130) and "Riemannian manifolds and conformal transformation groups" (Tensor, 8 (1958), 123–150). But the author found out that theorem 5.4 (p. 129) in the first paper and theorem 9, (c) (p. 141) in the second paper are incorrect. The purpose of this note is to not only correct these theorems but to prove the revised theorems.

1. First we consider theorem 9 (c) in the second paper. We shall use the same notations and terminologies as those of the original paper and the theorem 9 must be reviced as follows:

Let $\mathfrak{M}$ be a connected Riemannian manifold of dimension n and assume that $\mathfrak{M}$ admit an effective conformal transformation group G. If any one of the following three conditions holds, then $\mathfrak{M}$ is locally conformally flat.

(a) $n>2, \neq 4$ *and* $\frac{1}{2}\, n(n-1)+1<\dim. G<\frac{1}{2}\, n(n+1)+1$,

(b) $n>4, \neq 8$ *and* $\dim. G=\frac{1}{2}\, n(n-1)+1$,

(c) *Except for a finite number of n's*

$$\frac{1}{2}(n-1)(n-2)+3<\dim. G<\frac{1}{2}\, n(n-1)+1.$$

We shall prove the case (c). Take an arbitrary point p of $\mathfrak{M}$. Then when G_p° is isometric at p, we have, by using theorem 3 (p. 132),

$$\dim. \widetilde{M}_p=\dim. G_p-\dim. K_p\geqq\dim. G-n>\frac{1}{2}(n-2)(n-3)+1.$$

According to a theorem due to D. Montgomery and H. Samelson (Transformation groups of spheres, Ann. of Math., 44 (1943), 454–470), in an n-dimensional Euclidean space, except for a finite number of n's, the full rotation group has no subgroup H such that

$$\frac{1}{2}(n-1)(n-2)<\dim. H<\frac{1}{2}\, n(n-1)$$

or

$$\frac{1}{2}(n-2)(n-3)+1<\dim. H<\frac{1}{2}(n-1)(n-2).$$

It follows from this theorem that except for a finite number of n's $\dim. \widetilde{M}_p=\frac{1}{2}\, n(n-1)$ or $\dim. M_p=\frac{1}{2}(n-1)(n-2)$. When the former case holds, from theorem 4 (p. 136), if $n>3$ $C_p=0$ and if $n=3$ $C_p^{\circ}=0$. When the later case holds, from theorem 5

(p. 137), if $n>4, \neq 8$, $C_p=0$.

When G_p° is homothetic at the point p, from a theorem of S. Ishihara and M. Obata (On the group of conformal transformations of a Riemannian manifold, Proc. Japan Acad., 31 (1955), 426-429), if $n\geqq 3$ $C_p=0$ and if $n=3$ $C_p^{\circ}=0$. Gathering the above results, except for a finite number of n's, $\mathfrak{M}$ is locally conformally flat because the point p is arbitrary.

2. Next we consider theorem 5.4 in the first paper. Under the same notations and terminologies as those of the original paper, theorem 5.4 must be reviced as follows.

Except for a finite number of N's, if R_N for $N\geqq 3$ admits G_r of order r such that

$$\frac{1}{2}(N-1)(N-2)+3<r<\frac{1}{2}N(N-1)+1$$

then the R_N is conformally flat.

This theorem is proved in the same way as the above by using theorems 4.1 and 4.2 (p. 125) and the theorem of D. Montgomery and H. Samelson.

3. We list into the following table the corrections which are given rise from the above cited revisions of theorems 5.4 and 9, (c) and the corrections on some misprints.

On the first paper,

p. 114, line 27, For "$\frac{1}{2}(N-1)(N-2)+2$" read "$\frac{1}{2}(N-1)(N-2)+3$".

p. 125, line 2, For "g^{jf}" read "g^{hf}".

p. 129, line 7. For "$\frac{1}{2}(N-1)(N-2)+2<r$" read "$\frac{1}{2}(N-1)(N-2)+3<r$".

line 10, For "$r-N>\frac{1}{2}(N-2)(N-3)$" read "$r-N>\frac{1}{2}(N-2)(N-3)+1$".

line 15, For "$\frac{1}{2}(N-2)(N-3)<t$" read "$\frac{1}{2}(N-2)(N-3)+1<t$".

On the second paper,

p. 134, line 9, For "$\alpha(\varphi\Phi_t, p\Psi_h^{-1})=\alpha(\Phi_t\varphi, p)$ and $\alpha(\Psi_h\varphi, p)$".
read "$\alpha(\varphi\Phi_t, p\Psi_h^{-1})=\alpha(\varphi\Phi_t, p)$ and $\alpha(\varphi\Psi_h, p)$".

p. 137, line 14, For "$\tilde{m}$" read "$\mathfrak{M}$".

p. 142, line 1, For "$\frac{1}{2}(n-1)(n-2)+2<\dim. G$" read "$\frac{1}{2}(n-1)(n-2)+3<\dim. G$".

line 25, For "$\frac{1}{2}(n-2)(n-3)<\dim. H$" read "$\frac{1}{2}(n-2)(n-3)+1<\dim. H$".

p. 149, line 3, For "$\frac{1}{2}n(n+1)+1$" read "$\frac{1}{2}n(n-1)+1$".

ERRATA

A HYDROLOGICAL STUDY ON THE OLD CRATER OF ASO

By Toshisato MUROTA

Kumamoto Journal of Science A. Vol. 4, No. 1, (1959)

Page	Line	Wrong	Correct
39	10	ground water discharge	surface flow

Page	Place	Wrong	Correct
37	Table 3	12–13 Feb. 1444	12–13 Feb. 1941

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